Understanding ghosts in higher order theories of gravity

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General relativity (GR)

- 1915: GR : $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$
- One field: $g_{\mu\nu}$
- Observed predictions: GWs, Black Hole, Cosmological observables...
- This is our best description of gravity... for now !

$$S_{EH} = \int d^4x\sqrt{-g}\,R \tag{1}$$

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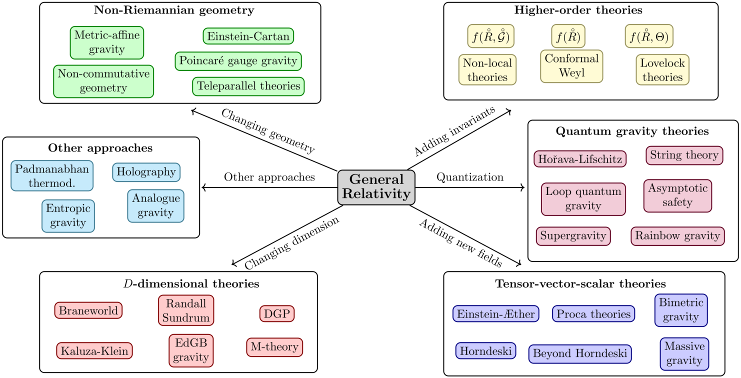
But some limits are known. Among them:

- Non renormalizable
- Singular solutions
- Cosmological tensions

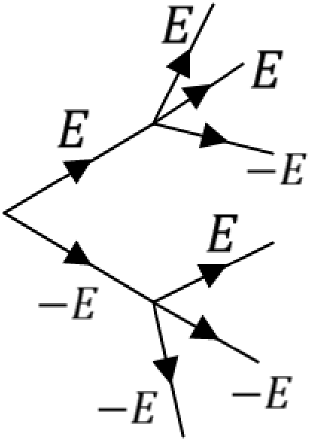
Going Beyond: How ?

Sebastian Bahamonde et al 2023 Rep. Prog. Phys. 86 026901



Ghosts instabilities

A classical and (especially) quantum instability, arising from an unbounded hamiltonian (from below and above).



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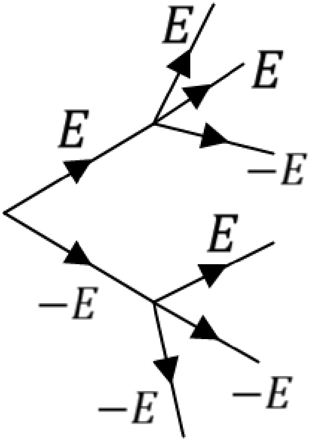

Ostrogradski theorem

Let $\mathcal{L}(q, ..., q^{(n)})$ be a **non-degenerate** lagrangian with $n > 1$, then the Hamiltonian is unbounded from below.

Consequence: presence of ghost, *i.e* a negative energy d.o.f.

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GR Lagrangian and Hamiltonian

$$\mathcal{L}_{EH} = \frac{1}{2\kappa c}\sqrt{-g}\ R \tag{2}$$

ADM decomposition:

$$\mathcal{L}_{EH} = \frac{1}{2\kappa c}\sqrt{-g}\ \left({}^{(3)}R + K_{ij}K^{ij} - K^2 - 2\nabla_\mu(a^\mu - n^\mu K)\right) \tag{3}$$

This leads to the so called ADM Hamiltonian:

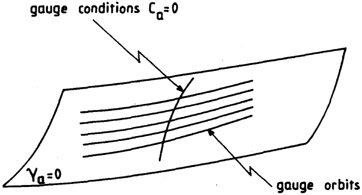
$$\mathcal{H} = \frac{1}{2\kappa c}(N\mathcal{H}_0 + N^i\mathcal{H}_i) \tag{4}$$

with

$$\boxed{\mathcal{H}_0 = \sqrt{\gamma}\left(\frac{1}{\gamma}(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2) - {}^{(3)}R\right)\ ;\ \mathcal{H}_i = -2D_j\pi_i^j} \tag{5}$$

Constraints and stability

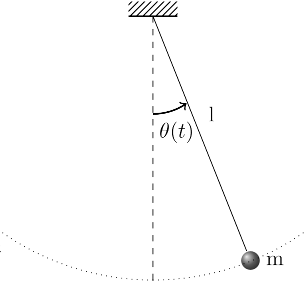
First class constraints



"Quantization of Gauge System", M.Henneaux & C.Teitelboim

- Algebraic relation between phase space variables.
- Generates gauge transformations.
- Translates a redundency in system's description.

Second class constraints



- Algebraic relation between phase space variables.
- Translates a "physical" constraint.

How to study f(R) stability

Method 1

$$\mathcal{L}_{MEH} = \sqrt{-g}\ f(R) \longrightarrow \mathcal{L}_{BD} = \sqrt{-g}\ (\phi R - V(\phi)) \quad (6)$$

$$\downarrow$$

$$\mathcal{L}_{BD} = \sqrt{-\tilde{g}}\ (\tilde{R} - \frac{1}{2}(\partial\tilde{\phi})^2 - \tilde{V}(\tilde{\phi})) \quad (7)$$

Method 2

Proceed Hamiltonian analysis : find all constraints and count number of propagating degrees of freedom. Study boundedness of the final hamiltonian if needed.

Second method is way more subtle

Crisostomi,Noui,Charmousis,Langlois arXiv:1710.04531 [hep-th]

Structure of $f(R)$

$$\mathcal{L}_{BD} = \sqrt{-g}\ (\phi R - V(\phi))$$

$$R = {}^{(3)}R + K_{ij}K^{ij} - K^2 - 2\nabla_\mu(a^\mu - n^\mu K) \qquad (8)$$

Canonical hamiltonian constraint H_0:

$$\mathcal{H}_0 = \frac{1}{\phi\sqrt{\gamma}}\left[\frac{1}{6}(\phi p_\phi - \pi)^2 - \pi^2 + \pi_{ij}^{TF}\pi_{TF}^{ij}\right] - \sqrt{\gamma}^{(3)}R\phi - 2\Delta\phi \qquad (9)$$

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Further more, it seems that any excitation of the volume give a safe theory:

$$\mathcal{L}_{MEH} = \sqrt{-g}\ \ (R + f(K)n^\mu\nabla_\mu\phi) \tag{10}$$

Physical meaning ?

Table des matières

Presentation of Gauss-Bonnet Gravity

The Gauss-Bonnet invariant is a scalar defined as:

$$G \triangleq (\star R^{\mu\nu}{}_{\alpha\beta})\, R^{\alpha\beta}{}_{\mu\nu}$$

In 4D, this scalar is purely topological.

$$G = \nabla_\mu \mathcal{J}^\mu$$

- Motivated by Lovelock gravity[1], ST[2]. A natural extension of GR.

1. Lovelock, D. (1971). "The Einstein tensor and its generalizations" doi:10.1063/1.1665613.
B. Zwiebach, Phys. Lett. B156 (1985) 315-317

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- Motivated by Lovelock gravity[1], ST[2]. A natural extension of GR.
- An expression exists for $\mathcal{J}^\mu$. A powerful way to study the structure of $R + f(G)$ models.

1. Lovelock, D. (1971). "The Einstein tensor and its generalizations" doi:10.1063/1.1665613.
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Toy model: anisotropic cosmology

Idée: Break isotropy to let the ghost (if any) appear.

Anisotropic ansatz

$$g_{\mu\nu} = \begin{pmatrix} -N^2 & 0 & 0 & 0 \\ 0 & a_1(t)^2 & 0 & 0 \\ 0 & 0 & a_2(t)^2 & 0 \\ 0 & 0 & 0 & a_3(t)^2 \end{pmatrix} \; ; \; H_i \triangleq \frac{\dot{a}_i}{a_i} \quad (11)$$

$$\mathcal{L} = \sqrt{-g}\,(R + \phi G\ ) \quad (12)$$

$$\downarrow$$

$$\tilde{\mathcal{L}} = \alpha \frac{V}{N}\,(H_1 H_2 + H_1 H_3 + H_2 H_3) + \beta \frac{V}{N^3} \dot{\phi} H_1 H_2 H_3 \quad (13)$$

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But, $H_1 H_2 H_3 = \frac{1}{6}\left(K^3 - K K^{ij} K_{ij} - 3 K_{ij} K^{ja} K_a^j\right)$. Now the "iso-volume" modes are excited !

Conclusion

- GB gravity is motivated by several approaches for quantum gravity.
- For higher order theories of gravity, constraints are needed to ensure the stability.
- $f(R)$ models only excite the volume.
- The anisotropic cosmology toy model can give a lot of informations concerning the excited modes. A deeper analysis is in process.

Conclusion

- GB gravity is motivated by several approaches for quantum gravity.
- For higher order theories of gravity, constraints are needed to ensure the stability.
- $f(R)$ models only excite the volume.
- The anisotropic cosmology toy model can give a lot of informations concerning the excited modes. A deeper analysis is in process.
- But presence of ghost is not sufficient to exclude a theory[1,2].

1. Deffayet, Held, Mukohyama, Vickman: ArXiv[2305.09631]
2. Andrei Smilga 2021 J. Phys.: Conf. Ser. 2038 012023