Positivity bounds on electromagnetic properties of media

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Not everything goes

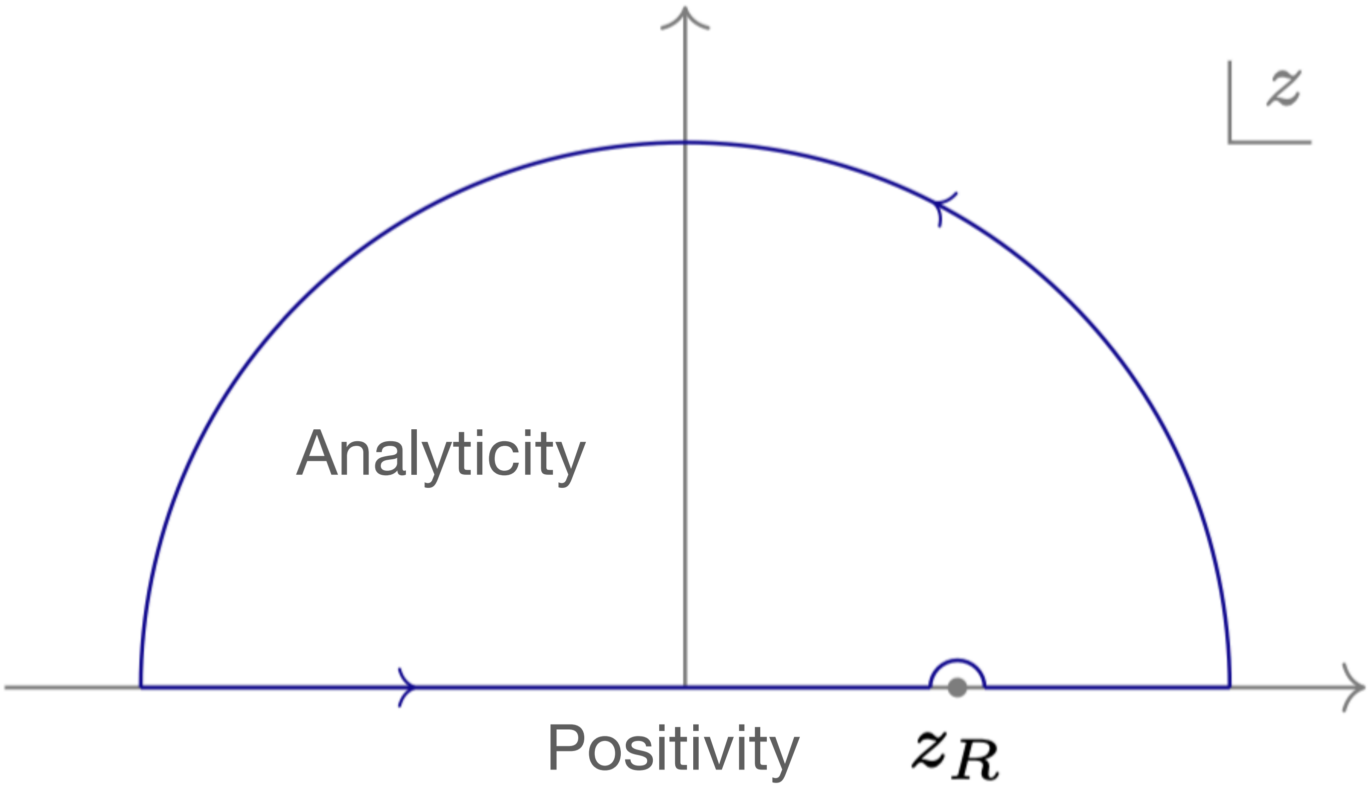
- Wilson coefficients are not arbitrary!
- E.g. $\mathscr{L} = -\frac{1}{4}F^2 + \frac{c}{\Lambda^4}F^4 + \ldots$ then $c > 0$.
- $2 \to 2$ scattering: Causality + Unitarity
- Known as S-matrix bootstrap, EFT positivity bounds

[Adams et. al. (2006)]

Breaking Lorentz

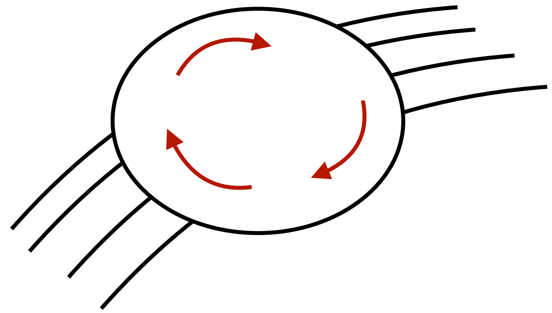
- Eg. Cosmology, Condensed matter, QFT @ finite T or Q
- S-matrix is no longer a suitable observable!
  - High and low energy states are no longer related by boost.
  - The amplitude is no longer analytic!
- Instead: Two point function of a conserved current $\langle J^\mu J^\nu \rangle$

[Creminelli et. al. (2022,2023), Hui et. al. (2023)]



We will see this in action for the case of EM response of media

Linear response



Application an external EM field $A^{\mu}_{\rm ext}$

$$J^{\mu}_{\rm in}(x) = \int {\rm d}^4y\, G^{\mu\nu}_J(x,y) A_{{\rm ext},\nu}(y)\,,$$ Kubo formula

In which the linear response is

$$G^{\mu\nu}_J(x,y) = i\theta(x^0 - y^0)\,\langle [J^\mu(x), J^\nu(y)]\rangle +$$ Contact terms

Retarded two point function

This object has two interesting properties: analyticity + positivity

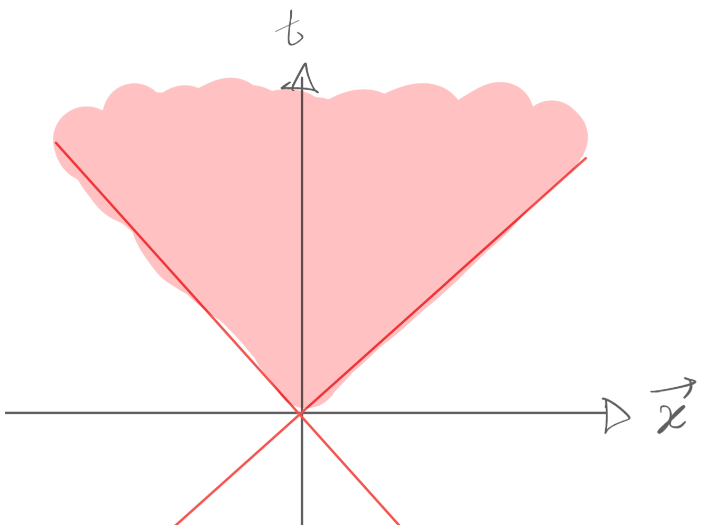
Causality → Analyticity

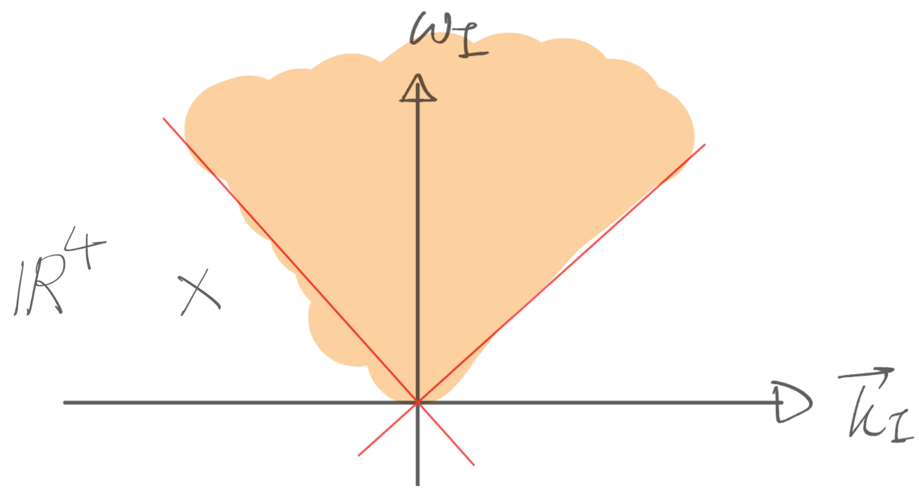
Linear response is retarded and micro-causal, therefore in Fourier space it is analytic in FLC

$\omega_I > |\vec{k}_I| > 0$

$$G_J^{\mu\nu}(p) = \int \mathrm{d}^4x\, e^{-ip\cdot x} G_J^{\mu\nu}(x)\,,$$

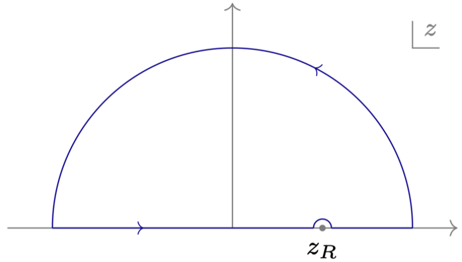
$\exp(-\omega_I t + \vec{k}_I \cdot \vec{x})$





Leontovich

We can parametrize $\mathbb{C}^4$ as $p^\mu = (\omega, \boldsymbol{q} + \omega \boldsymbol{\xi})$ FLC $\rightarrow \xi < 1$



Maps the region of analyticity to the UHP

We can treat the analytic functions as being single variable and use Cauchy theorem, to derive Leontovich relation

χ analytic

$$\chi(\omega, \boldsymbol{q} + \omega \boldsymbol{\xi}) = \frac{1}{i\pi} \mathrm{PV} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{z - \omega} \chi(z, \boldsymbol{q} + z \boldsymbol{\xi})$$

Kramers-Kronig with ξ dependence

This relates the real part and the imaginary part of the function

Important assumption: The arc at infinity is negligible!

[Leontovich (1961), Creminelli et. al. (2022)]

Positivity

The imaginary part of the response function characterizes the dissipation of the energy of the external source in the system.

$$\Delta H = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \omega A_{\mathrm{ext},\mu}(-p)\, \mathrm{Im}\, G_J^{\mu\nu}(p) A_{\mathrm{ext},\nu}(p)\,,$$

This does not necessarily have a definite sign! Consider for example a Laser.

For passive medium it is positive. For these systems $\omega \mathrm{Im} G_J^{\mu\nu}$ is positive definite.

Resolving the imaginary part

After some algebra we can see that $\mathrm{Im} G_J^{\mu\nu}(p) \;\; = \frac{1}{2} \int \mathrm{d}^4 x \, e^{-ip\cdot x} \, \langle [J^\mu(x), J^\nu(0)] \rangle$

Inserting the resolution of the identity, with $\rho \, |n\rangle = c_n \, |n\rangle$, gives
density matrix

$$V_\mu(p) V_\nu(p) \, \mathrm{Im}\, G_J^{\mu\nu}(p) = \frac{1}{2} \sum_{n,m} (2\pi)^4 \delta(p + p_n - p_m) | \langle n | \, J^\mu(0) \, | m \rangle \, V_\mu(p) |^2 (c_n - c_m) \, .$$

Sign of the imaginary part depends on the state.

A sufficient condition for positive definite $\omega \mathrm{Im} G_J^{\mu\nu}$ is monotonicity of c_n

e.g. ground state, thermal state

No gap !

Action for photon in matter

The system is described by in-in effective action $\Gamma[A_1, A_2]$.

The usual in-out effective action is not causal in dissipative systems.

At quadratic order, the effect of matter is encoded in the photon self-energy

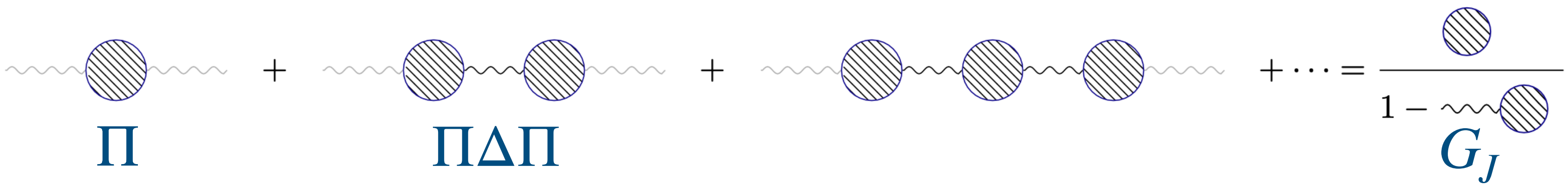
$$\Pi^{\mu\nu}(x,y) = i\theta(x^0 - y^0)\langle[J^\mu(x), J^\nu(y)]\rangle_{1\text{PI}} + \text{Contact terms}$$

In Fourier space

birefringence

$$\Pi^{\mu\nu} = -\pi_L(\omega,k)p^2\mathcal{P}_L^{\mu\nu} + \pi_T(\omega,k)k^2\mathcal{P}_T^{\mu\nu} + \text{Parity Violation}$$

Projection along k^i Projection orthogonal to k^i



Action for photon in matter

These are ε μ in disguise $\quad \varepsilon - 1 = g^2 \pi_L\,, \qquad 1 - \dfrac{1}{\mu} = g^2 \left(\pi_T - \dfrac{\omega^2}{k^2}\pi_L\right)$

The in-in effective action for photon is

$$\Gamma[A_r, A_a] = \frac{1}{g^2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\left[\varepsilon(\omega,k)\boldsymbol{E}_a(-p)\cdot\boldsymbol{E}_r(p) - \frac{1}{\mu(\omega,k)}\boldsymbol{B}_a(-p)\cdot\boldsymbol{B}_r(p) + \ldots\right]$$

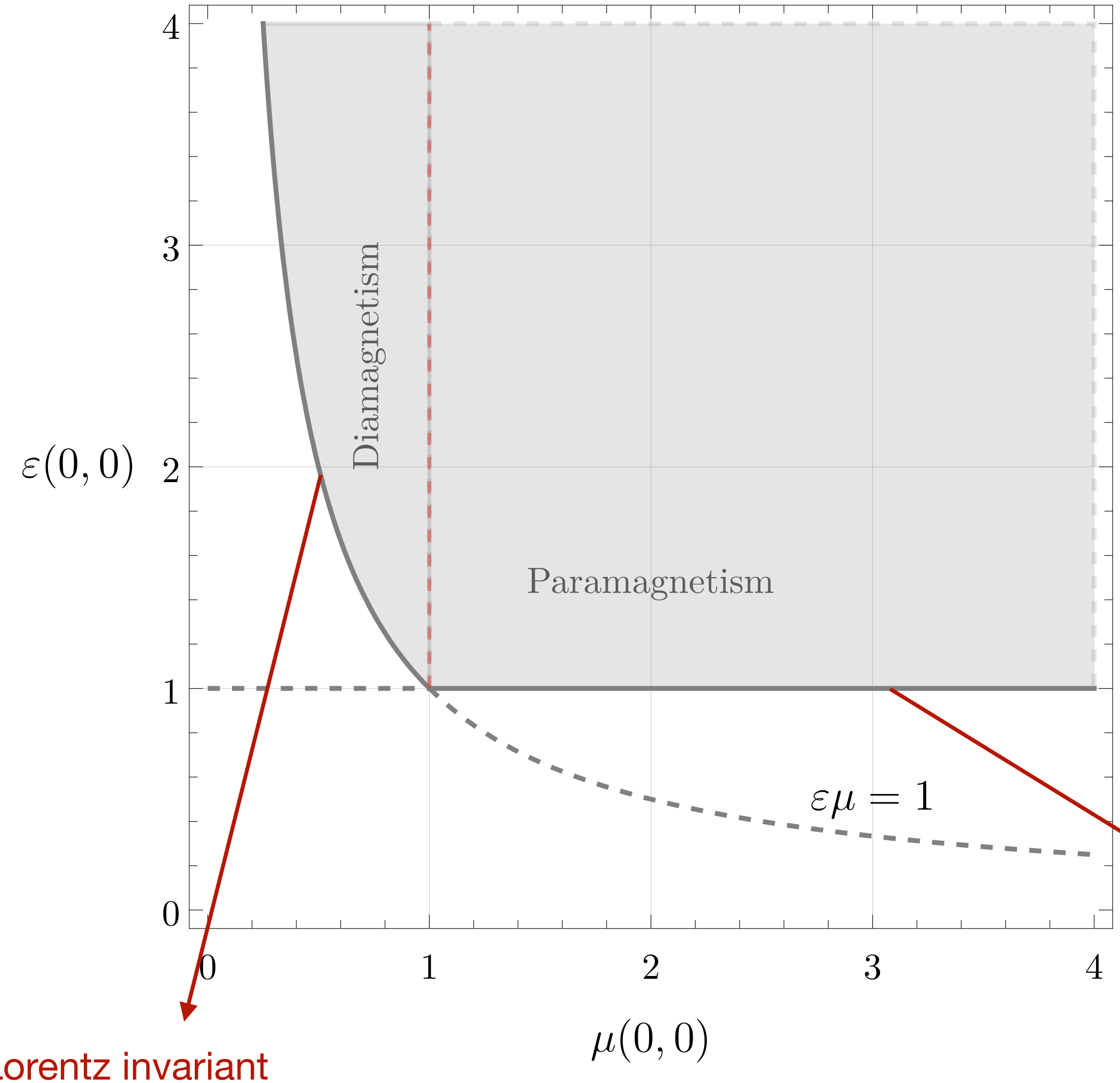
The low energy effective action is obtained by expanding the coefficients in powers of ω and k. For Insulators: no other relevant degrees of freedom, a local action: $\varepsilon = \varepsilon(0{,}0) + \ldots$

$$\varepsilon(0,0) - 1 = \frac{2g^2}{\pi}\int_0^{+\infty}\frac{\mathrm{d}z}{z}\;\mathrm{Im}\,\pi_L(z, z\boldsymbol{\xi})$$

$$(\varepsilon(0,0)-1) + \xi^2\left(1 - \frac{1}{\mu(0,0)}\right) = \frac{2g^2\xi^2}{\pi}\int_0^{+\infty}\frac{\mathrm{d}z}{z}\;\mathrm{Im}\,\pi_T(z, z\boldsymbol{\xi})$$

$$> 0$$

Transverse $\xi \to 1$, not super-luminal



Longitudinal, no para-electric

Purely magnetic response

$$g^2 \pi_T(\omega, \boldsymbol{k}) = \frac{\omega_T^2}{-\omega^2 - i\gamma\omega + c_s^2 k^2 + \omega_0^2}$$

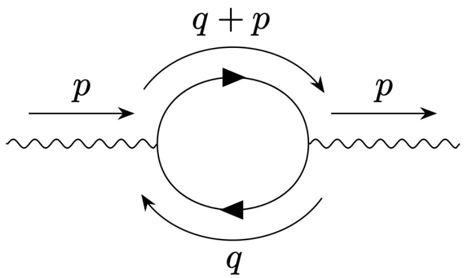
Lorentz invariant

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\alpha}{\Lambda^2}\phi^2 F_{\mu\nu}F^{\mu\nu}$$

High energy

The assumption is that the infinity arc can be neglected.

No Froissart bound for correlation functions!



A model for the medium at high energies: Free Fermi gas.

The response is known as the Lindhard function: similar to QED at finite chemical potential.

Plasma frequency: $\omega_p^2 = g^2 n/m$

High energy limit of the Lindhard function: $g^2\pi_L \to -\frac{\omega_p^2}{\omega^2} + \dots\,, \qquad g^2\pi_T \to -\frac{\omega_p^2}{k^2} + \dots\,,$

There is also a relativistic piece that grows logarithmically $\sim \log(p^2)$ starting from the pair production threshold $\sim m$.

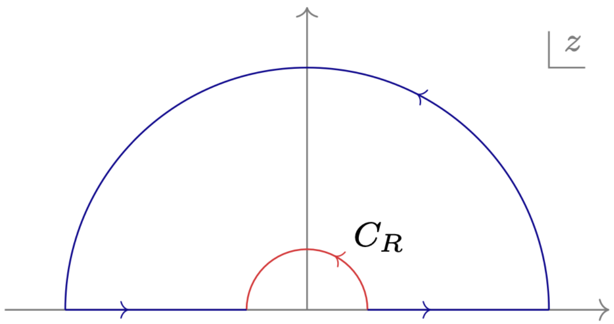
Typically, $\omega_p^2/m^2 \sim 10^{-12}$, so this is negligible if we close the contour at much smaller energies.

[Lindhard (1954)]

Higher Derivatives

Usually, the more interesting bounds appear for higher dimension operators in the EFT.

The situation gets complicated due to dissipation in the system.



$$\chi(z) \simeq \chi(0) + \chi'(0) z + \chi''(0) z^2/2 + \dots$$

Arc variable

$$\int_{C_R} \mathrm{d}z\, \frac{\chi(z)}{z^3} = 2i \int_R^{+\infty} \mathrm{d}z\, \frac{\mathrm{Im}\,\chi(z)}{z^3}$$

$$\chi''(0) = \frac{4}{\pi}\left(-\frac{\chi'(0)/i}{R} + \int_R^{+\infty} \mathrm{d}z\, \frac{\mathrm{Im}\,\chi(z)}{z^3}\right)$$

Negative

Positive

All dissipative terms actually contribute

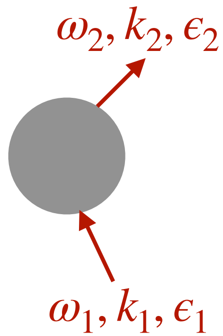
[Bellazzini, et. al. (2020)]

Compton scattering

We can think of a medium constructed out of a gas of (neutral) particles.

In the dilute limit, $\varepsilon \approx 1 + n\alpha$ and $\mu \approx 1 + n\beta$ in which α and β are electric and magnetic polarizability coefficients.

Polarizabilites can be measured by doing a low energy Compton scattering



$$\mathscr{M} \propto \left(\alpha\omega_1\omega_2\vec{\epsilon}_1 \cdot \vec{\epsilon}_2 + \beta(\vec{k}_1 \times \vec{\epsilon}_1) \cdot (\vec{k}_2 \times \vec{\epsilon}_2)\right)$$

Using the S-matrix positivity $\quad \alpha + \beta = \dfrac{1}{2\pi^2}\displaystyle\int \frac{\mathrm{d}\omega}{\omega^2}\sigma_{\mathrm{tot}}(\omega) > 0$

This is consistent with $\quad n(\alpha + \beta) \approx \varepsilon - \dfrac{1}{\mu} = \dfrac{2g^2}{\pi}\displaystyle\int \frac{\mathrm{d}z}{z}\mathrm{Im}\pi_T(z, z) > 0$

Gravity

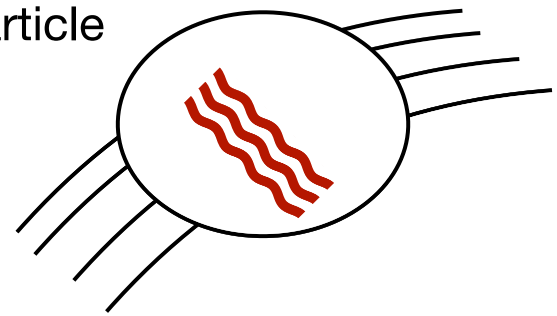
The analog of polarizability coefficient for gravity is know as tidal Love number.

Quantifies the response of a compact object to an external gravitational potential: electric and magnetic type.

They appear as coefficients in the low energy expansion of the two point function of the stress-tensor.

At least when gravity can be treated linearly, e.g. small compactness, the above analysis should apply: $\lambda_E > 0$, and $\lambda_E + \lambda_B > 0$ (And higher multipoles).

One can think of this as positivity bound for the point particle effective field theory



Summary

We studied bounds on the low energy response of a medium from causality and positivity.

- More general class of media, e.g. super-conductors
- Transport coefficients in a fluid
- Källén–Lehmann without Lorentz
- Cosmology

Thank you