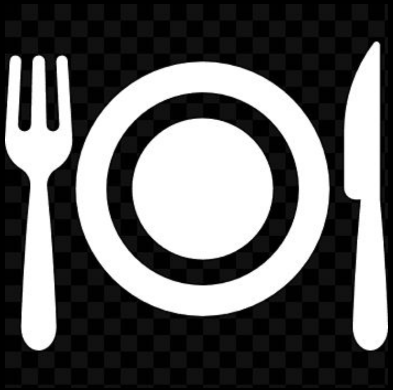
Czech Academy of Sciences

PRAGUE SPRING 2025:
CAS - IBS CTPU-CGA - ISCT
WORKSHOP
IN COSMOLOGY,
GRAVITATION AND PARTICLE
PHYSICS

Virial identities across the spacetime

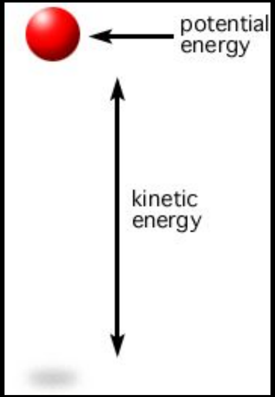
Alexandre M. Pombo

Introduction



Introduction: Virial Theorem

- The virial theorem relates the **average** kinetic and potential energy
- It allows the average kinetic energy to be calculated even for very complicated systems
- The theorem has found applications in several areas



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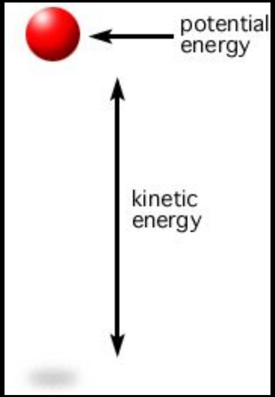

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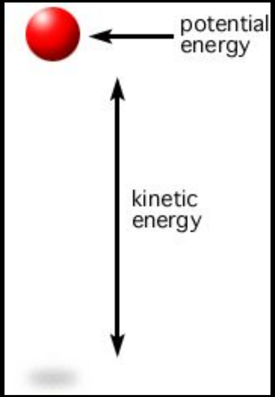

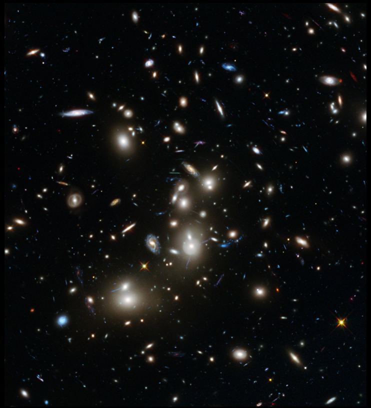

Virial identity

- Integral identities that are virial-like
- In field theory rather than particle mechanics
- It is obtained from scaling arguments
- Computed independently from the equations of motion
- Applicable to stationary spacetimes
- We present an approach for curved spacetimes

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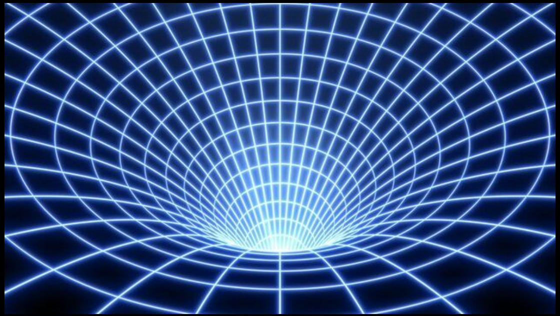

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Virial Identity

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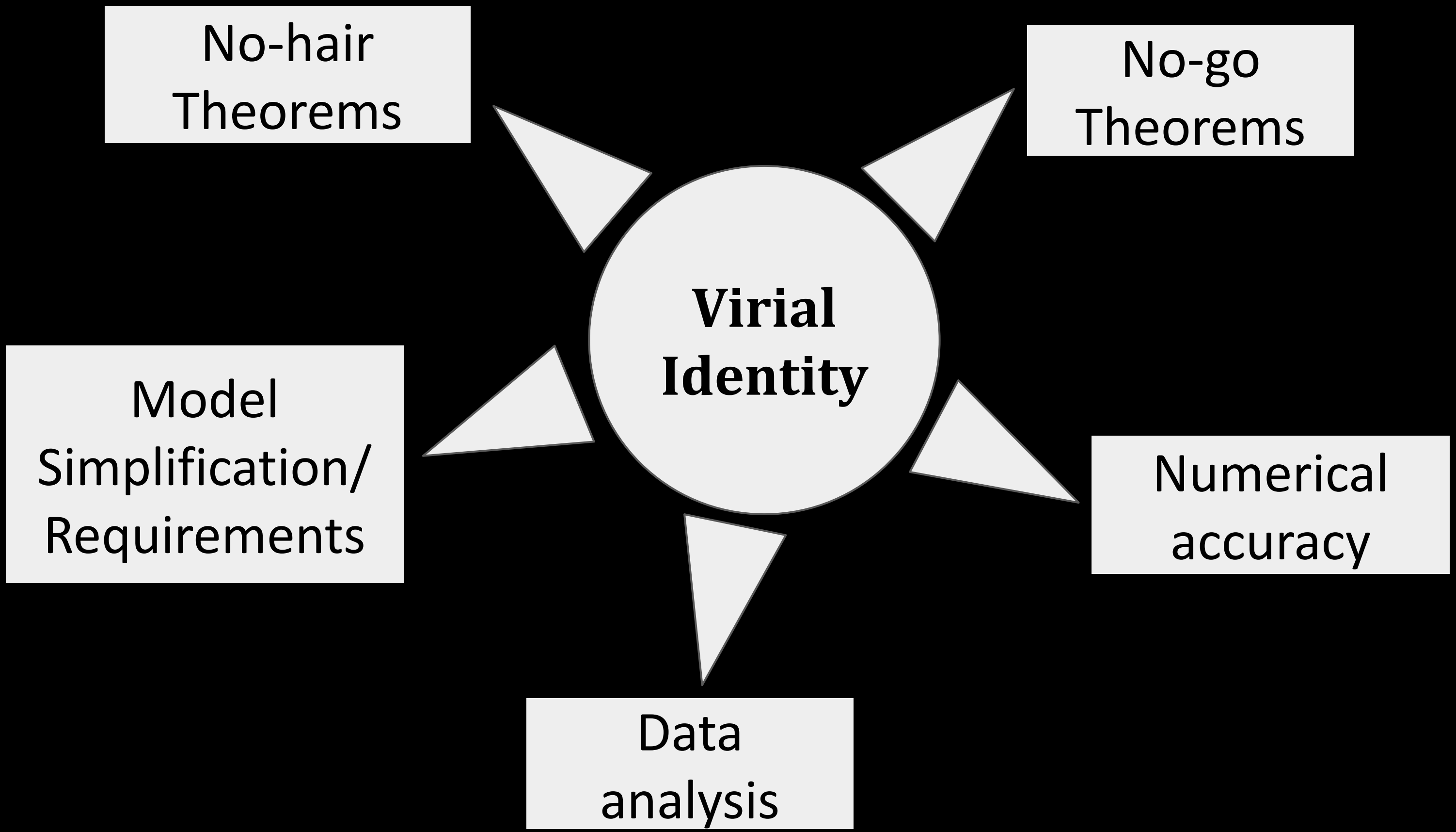
No-hair Theorems
No-go Theorems
Virial Identity
Model Simplification/ Requirements
Numerical accuracy
Data analysis

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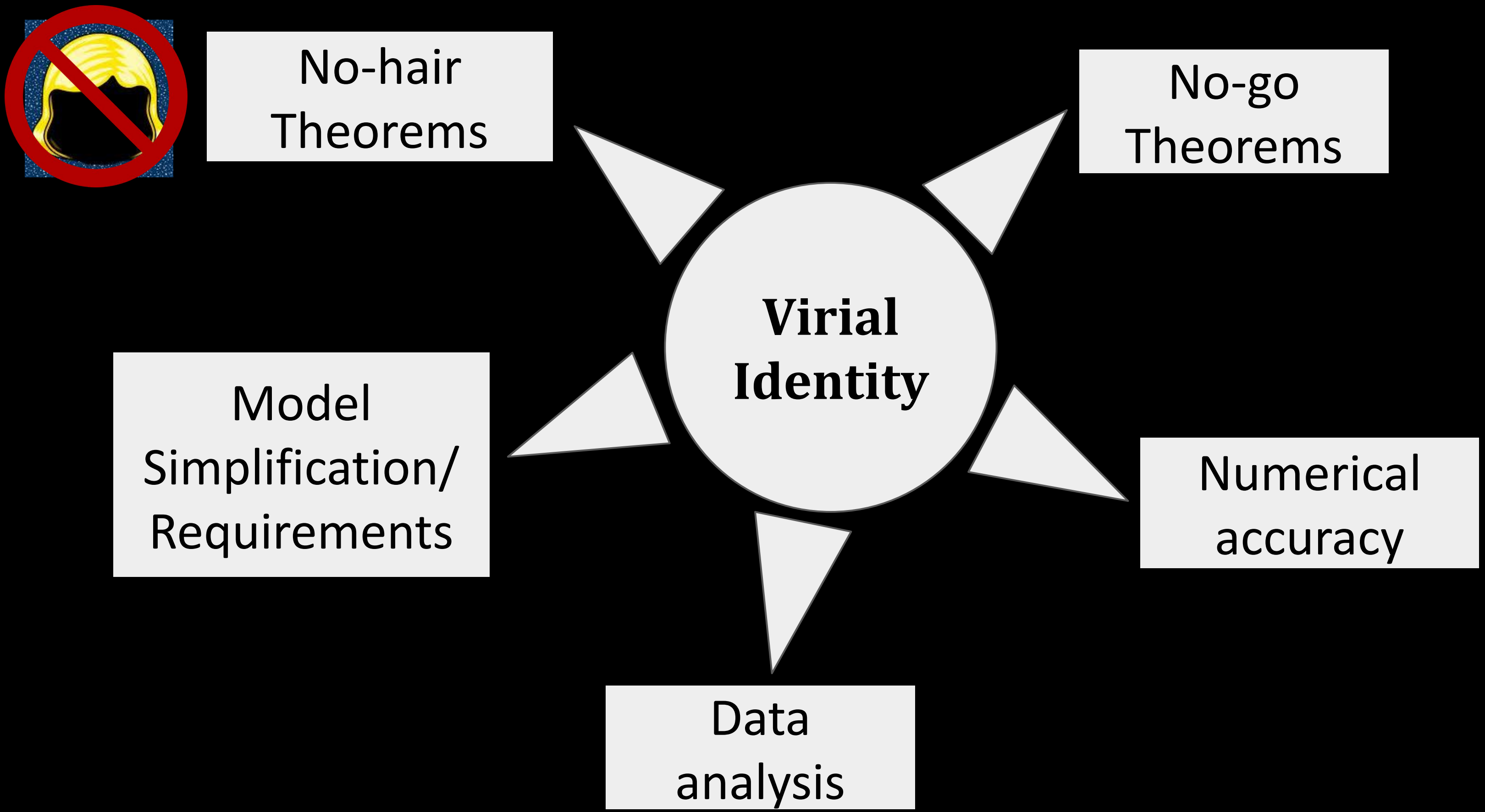
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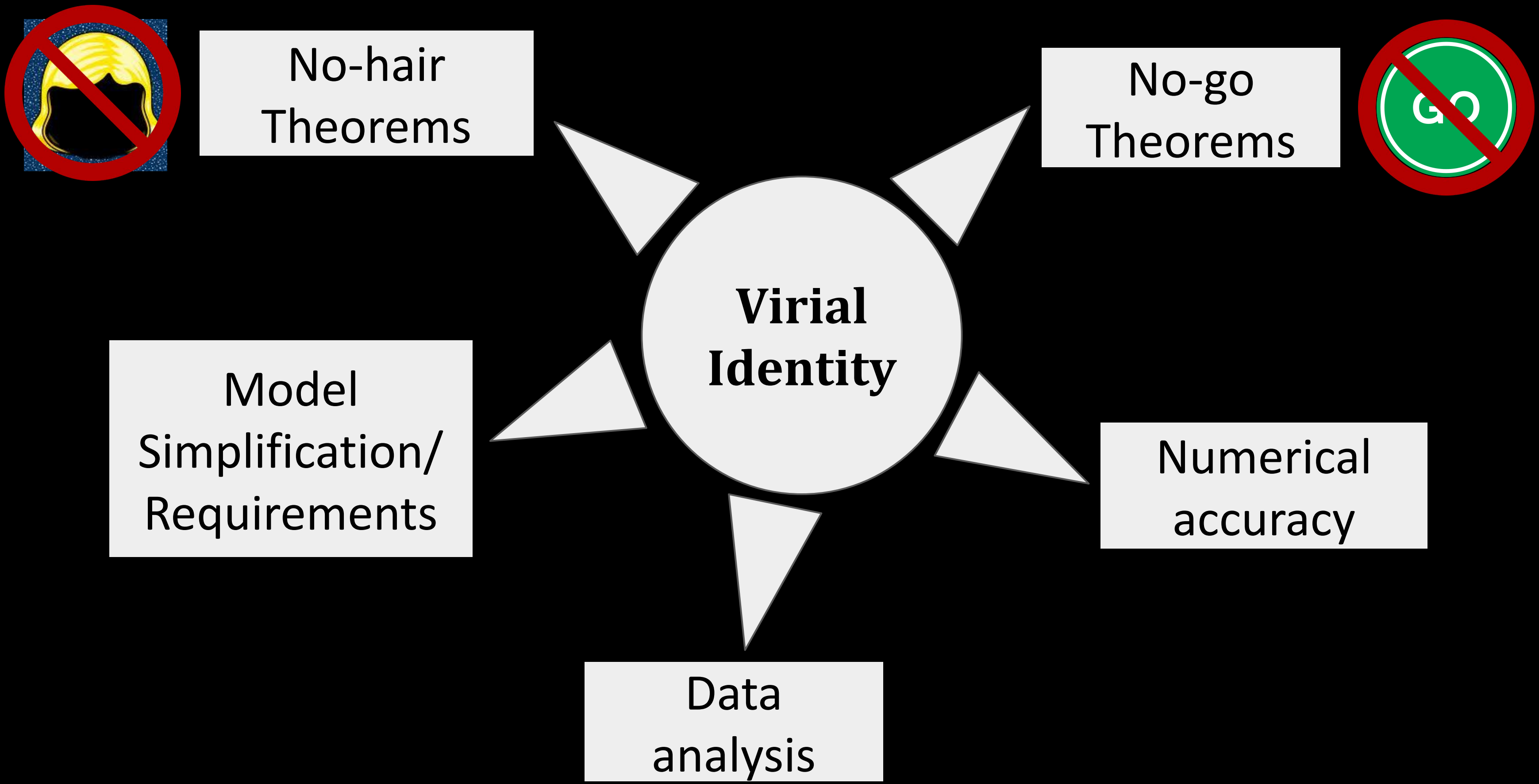
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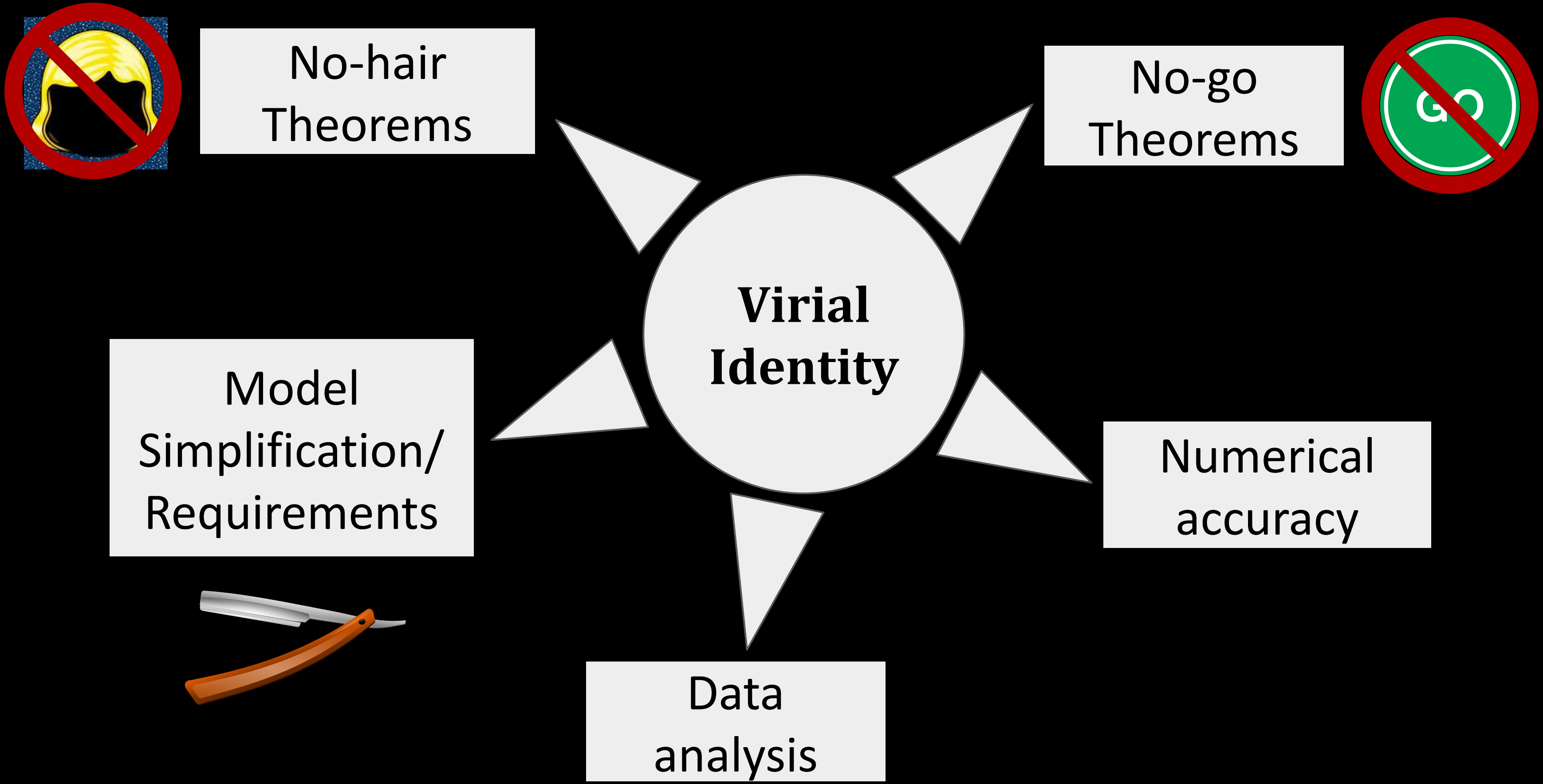
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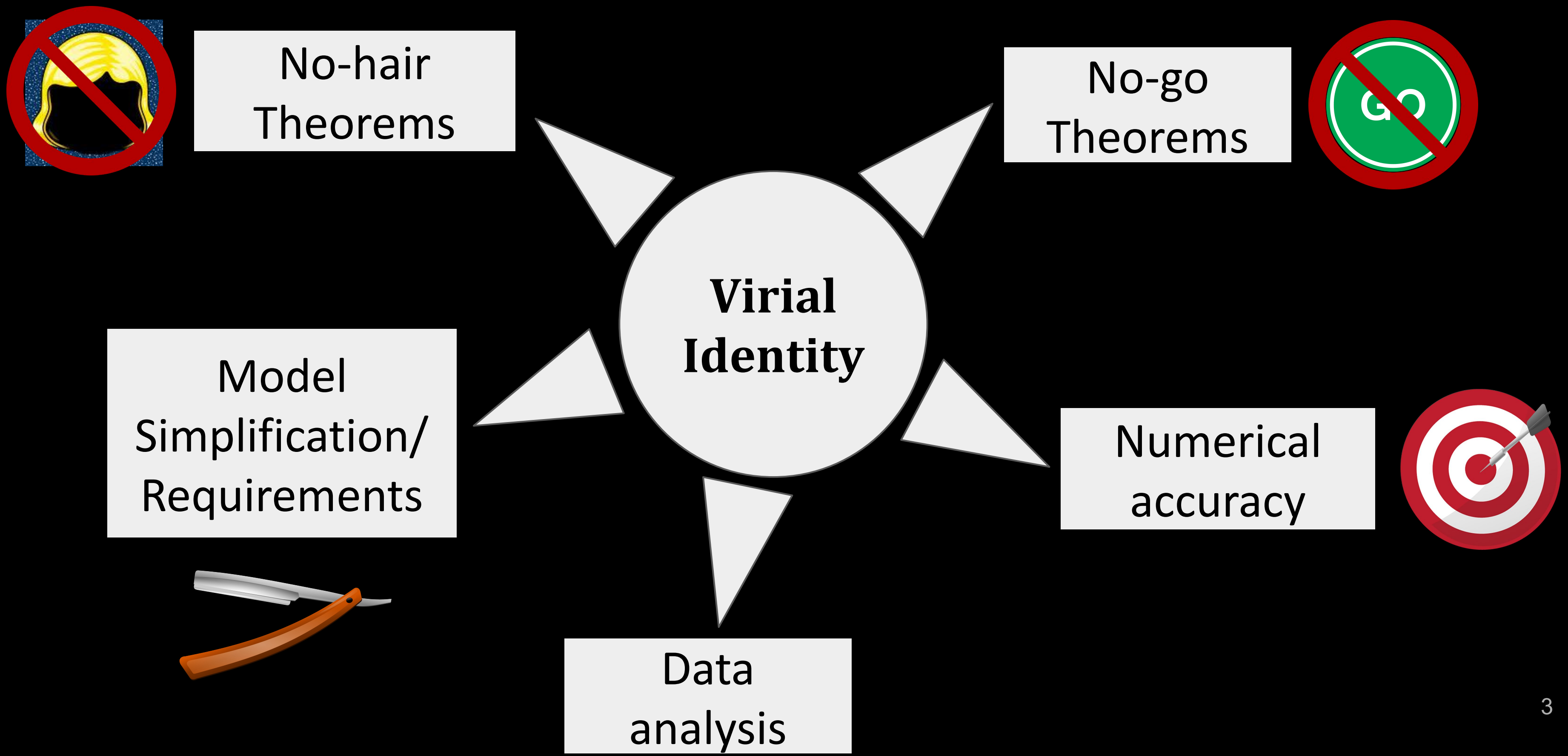
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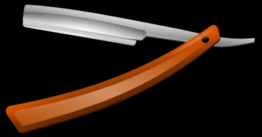

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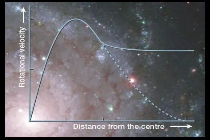
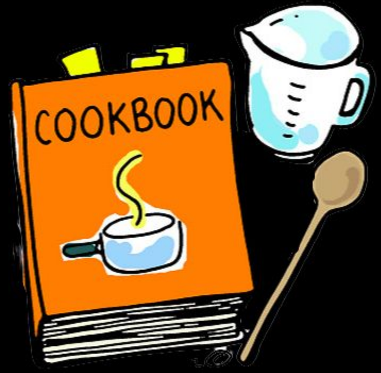


Recipe



Recipe

Ingredients:

- Action S
- Metric ansatz $g_{\mu\nu}$
- Matter ansatz
- Gibbons-Hawking-York term (gravity)

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Material:

- Derrick's scaling argument
- Hamilton's principle
- Love and patience

Step-by-step:

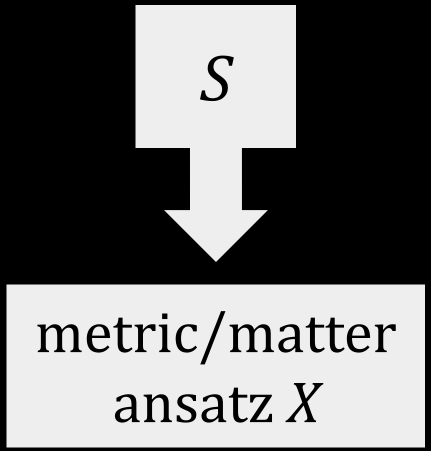
Step-by-step: Derrick's scaling argument

Model

$$S$$

Step-by-step: Derrick's scaling argument

Model



Step-by-step: Derrick's scaling argument

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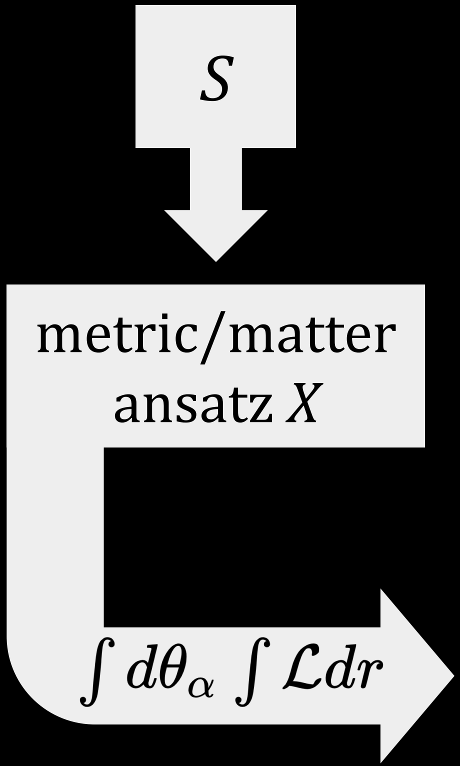

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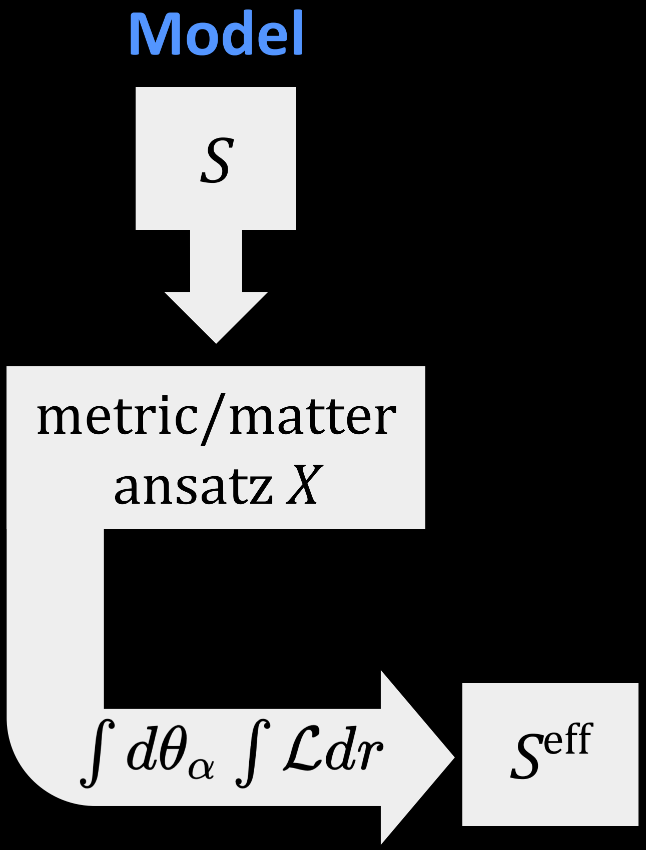

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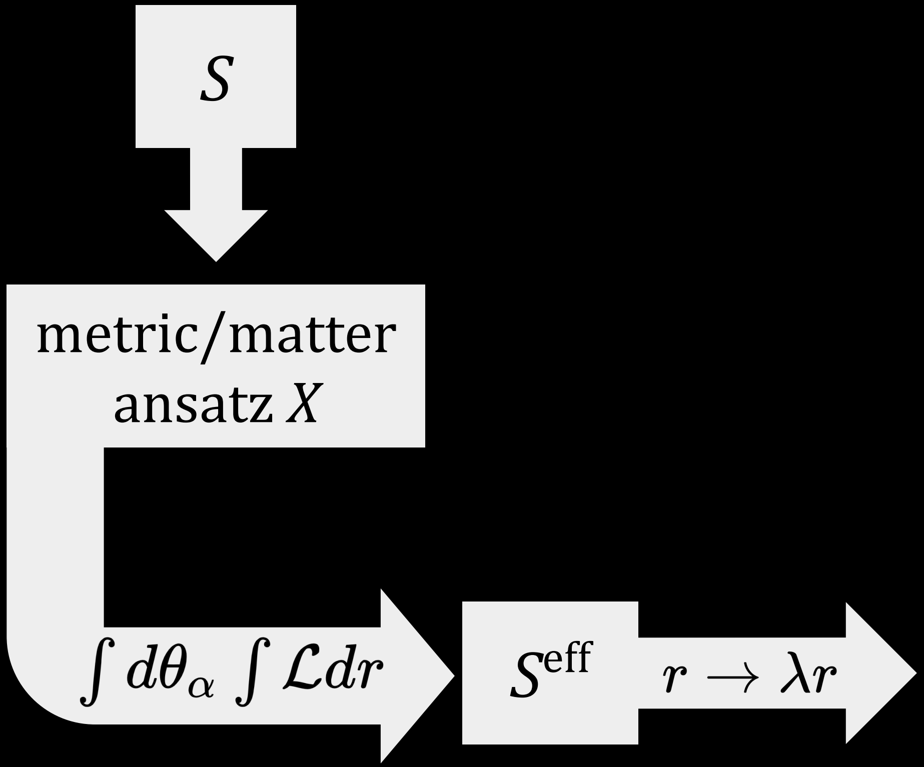

Step-by-step: Derrick’s scaling argument

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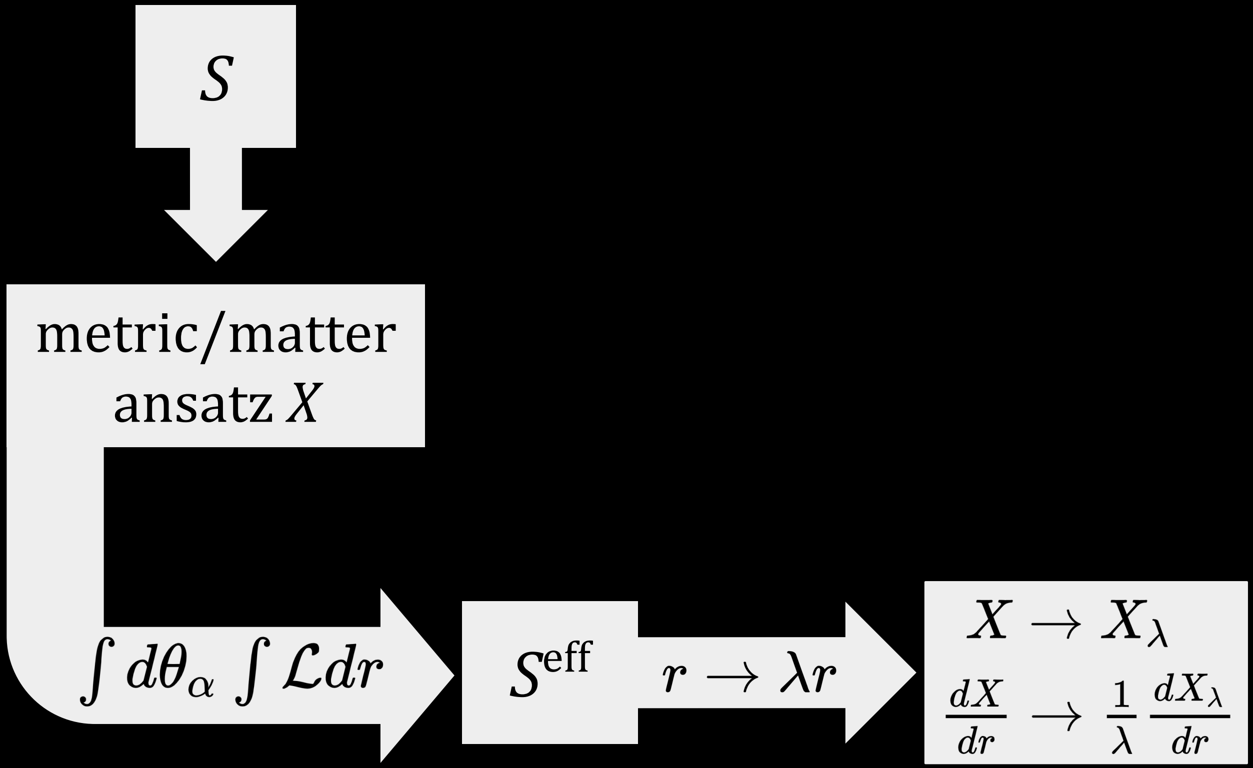

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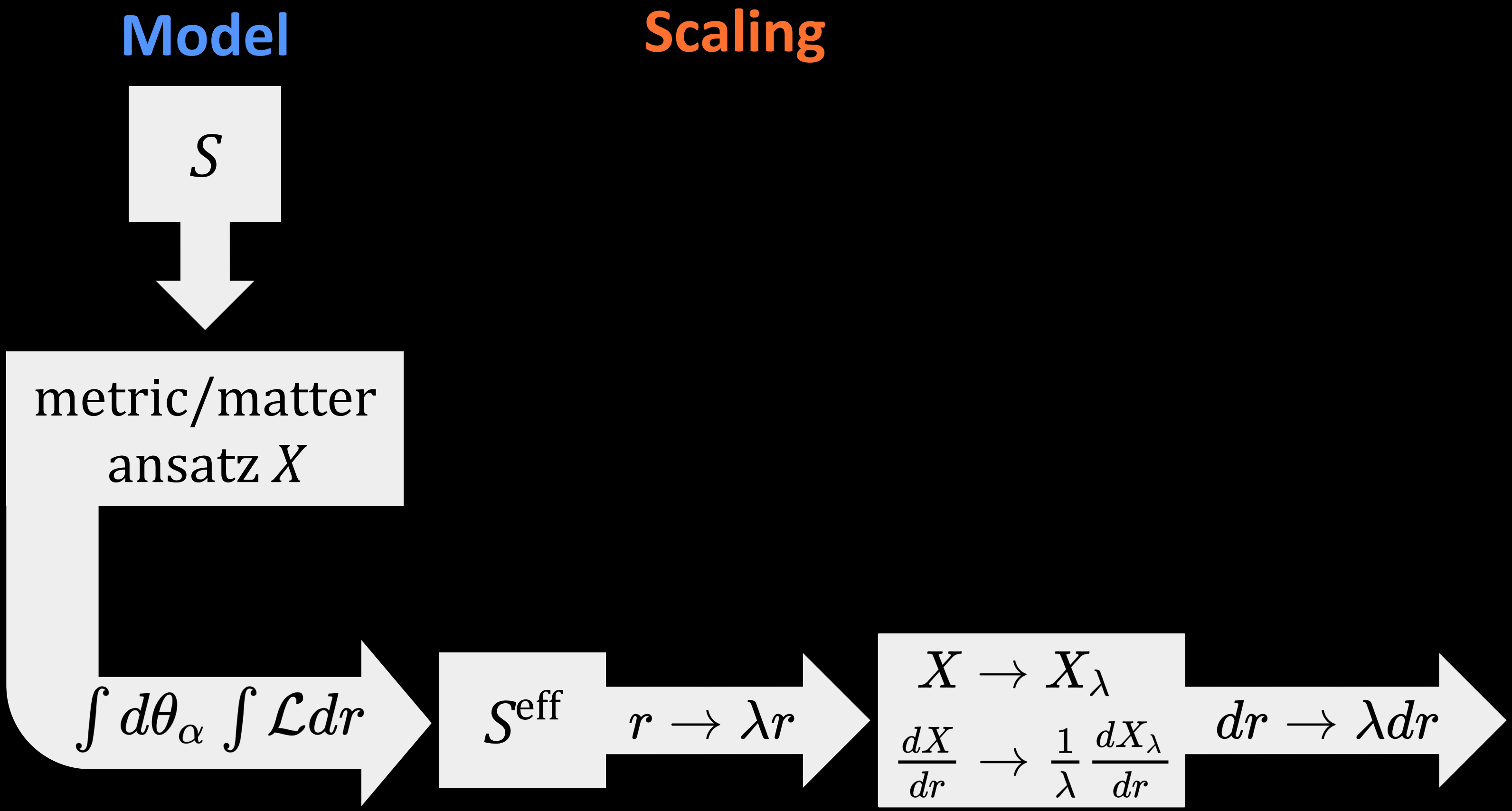

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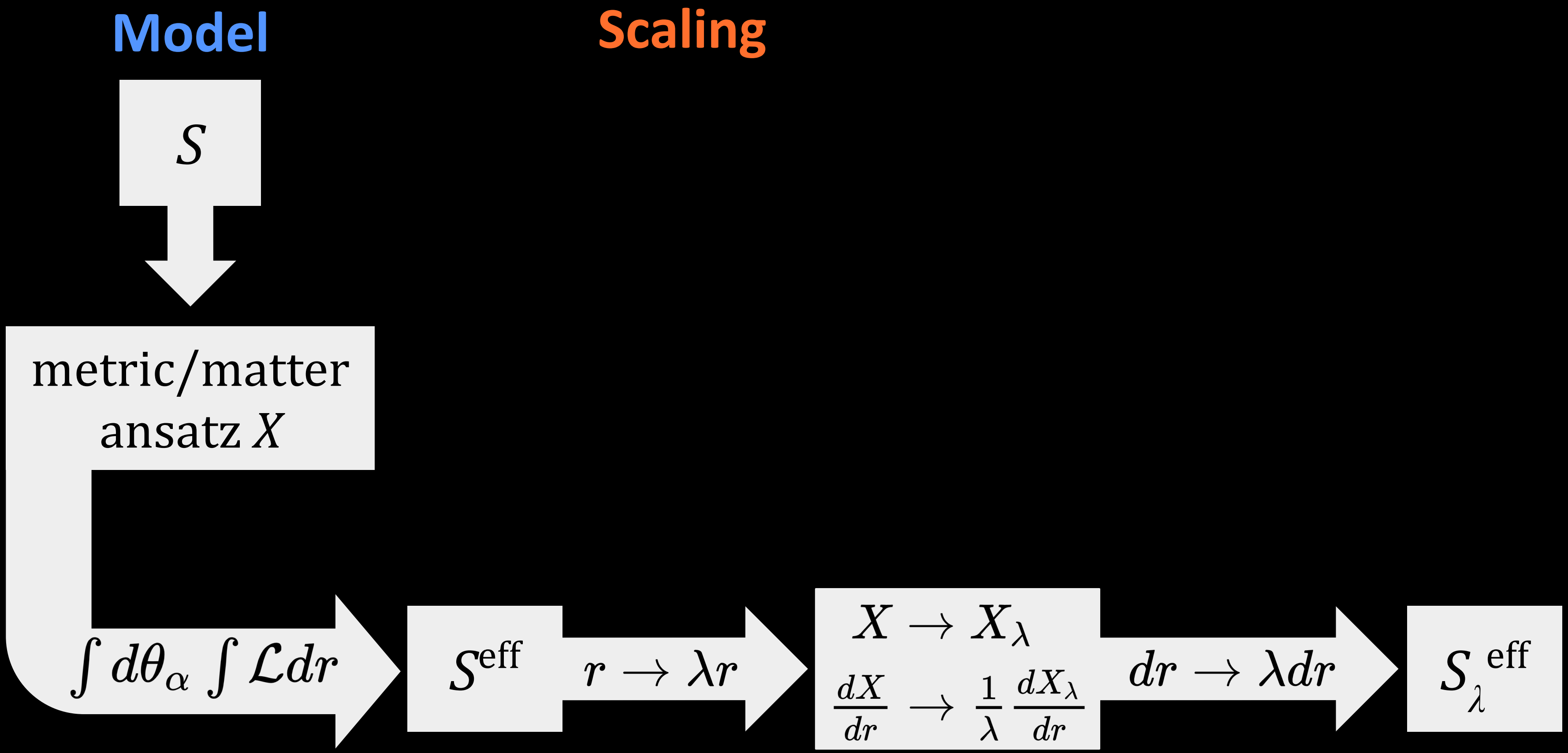

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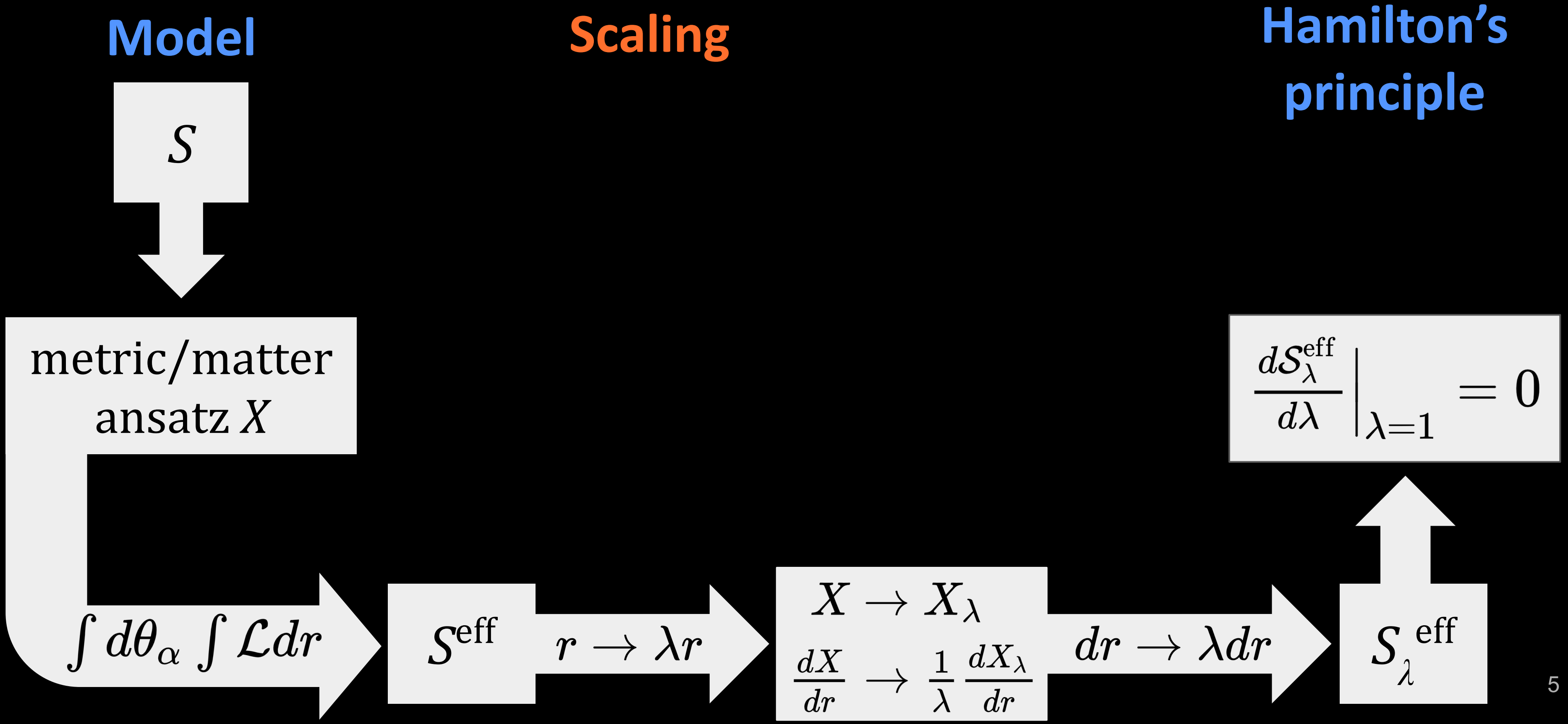

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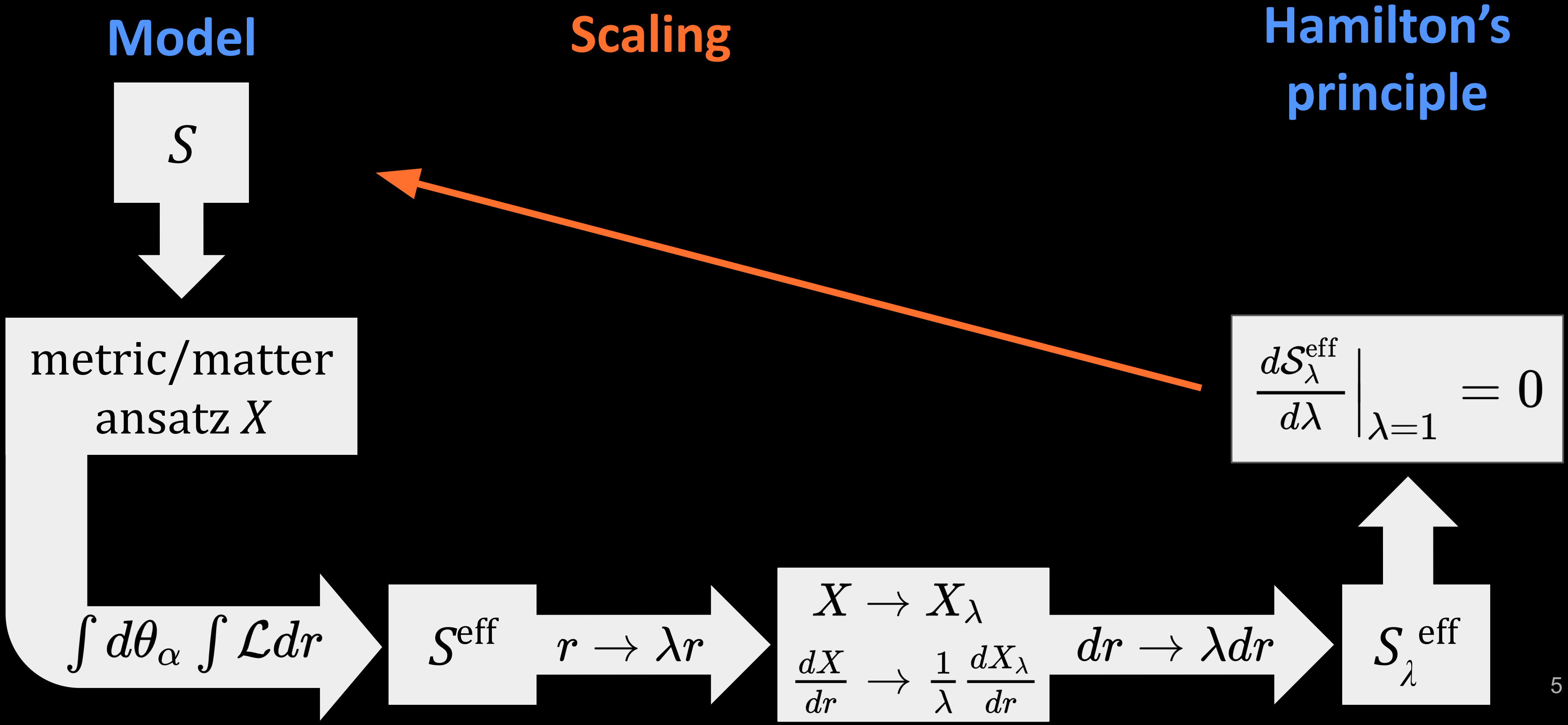

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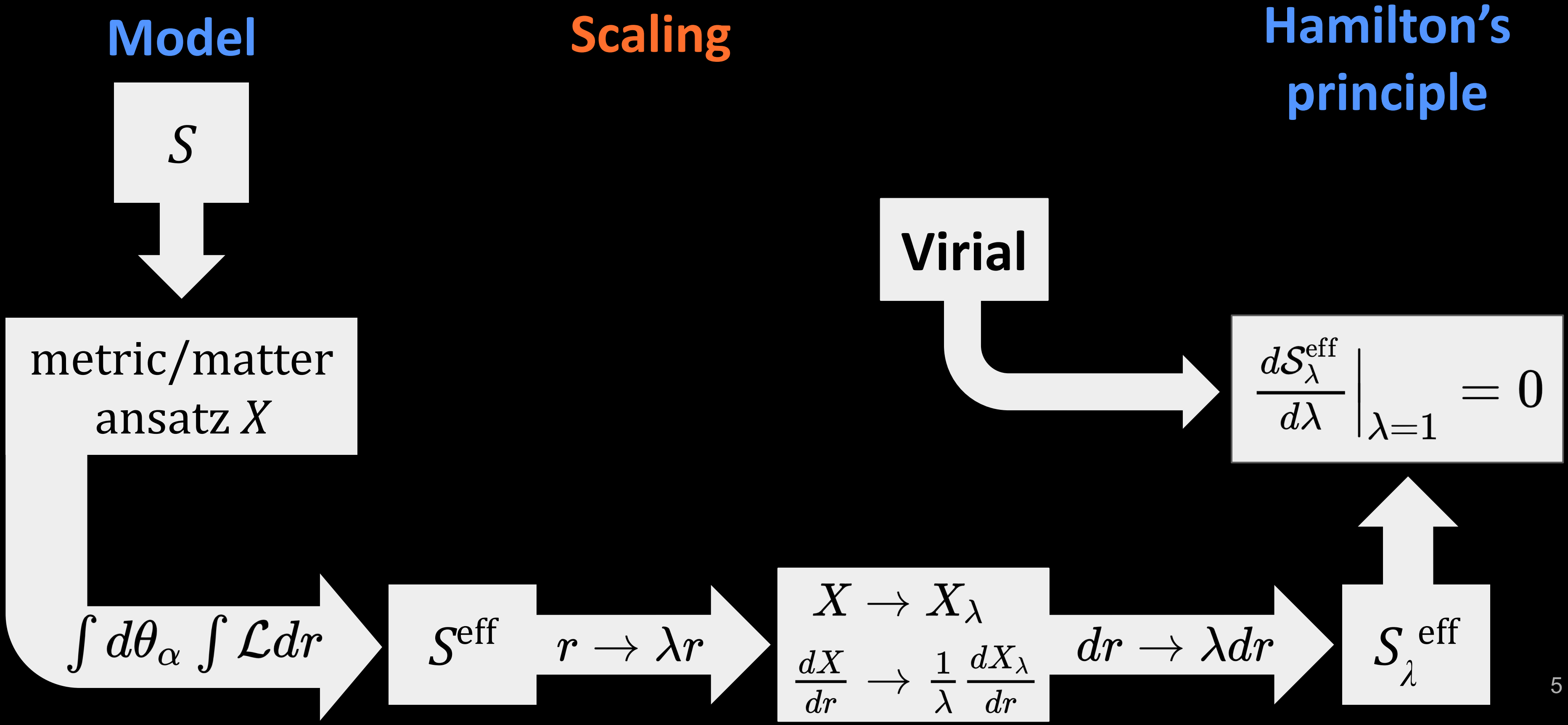

First dish

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Spherical Symmetry

PRD (104) 2021

Derrick's argument

- The action of a real scalar field

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- The scaled configuration

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Derrick's argument: Q-balls

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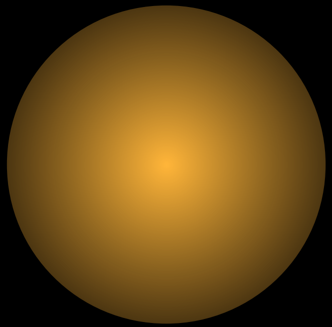

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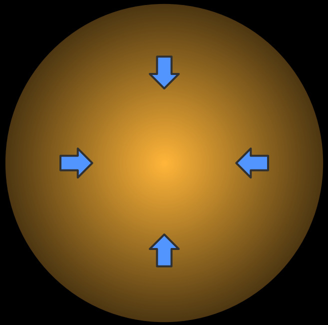

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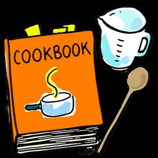

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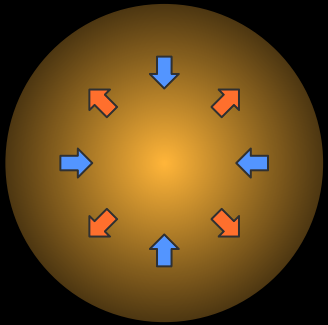

Derrick's argument: Gravity

- The gravitational action

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$$= \frac{1}{4}\int_{\mathcal{M}} d^4x\,\sqrt{-g}R + \frac{1}{2}\int_{\partial\mathcal{M}} d^3x\,\sqrt{-\gamma}(K - K_0)$$

r = 0

r = +∞

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The boundary term is needed since the gravitational Lagrangian density, *R*, contains second order derivatives of the metric tensor

Derrick's argument: Gibbons-Hawking-York

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$$ds^2 = -\sigma(r)^2 N(r) dt^2 + \frac{dr^2}{N(r)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

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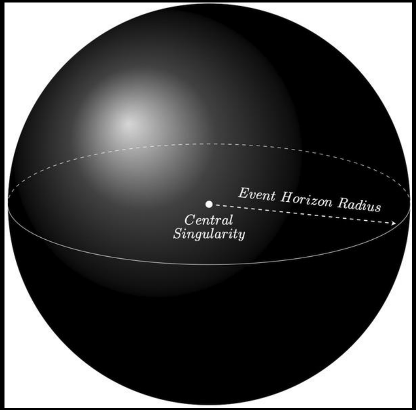
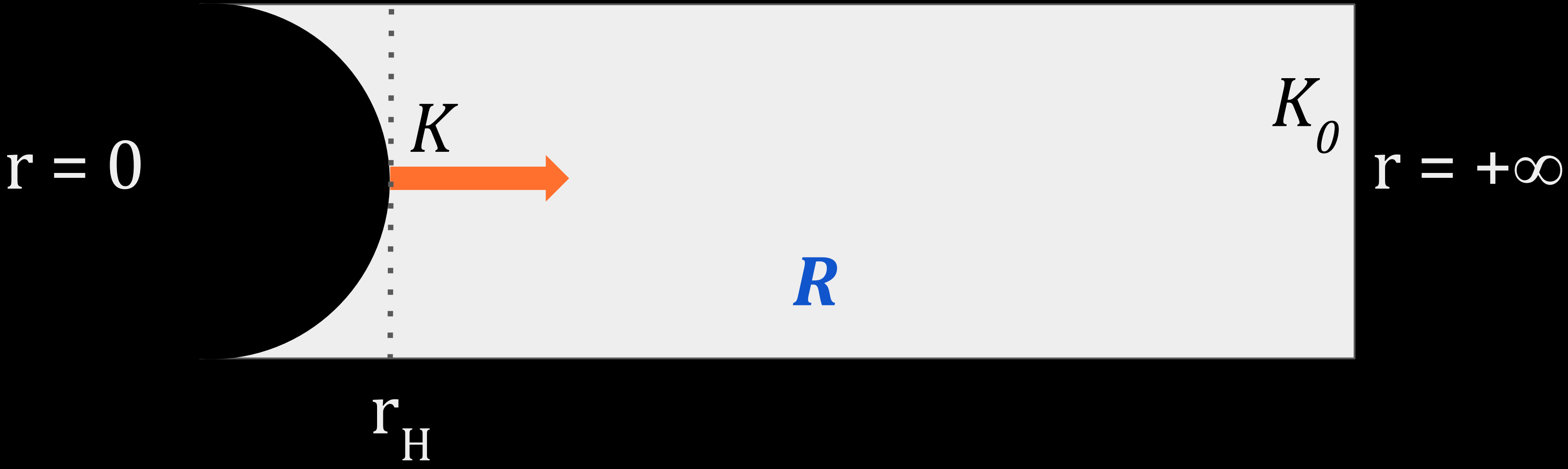


Derrick's argument: Black hole

- In the presence of an horizon



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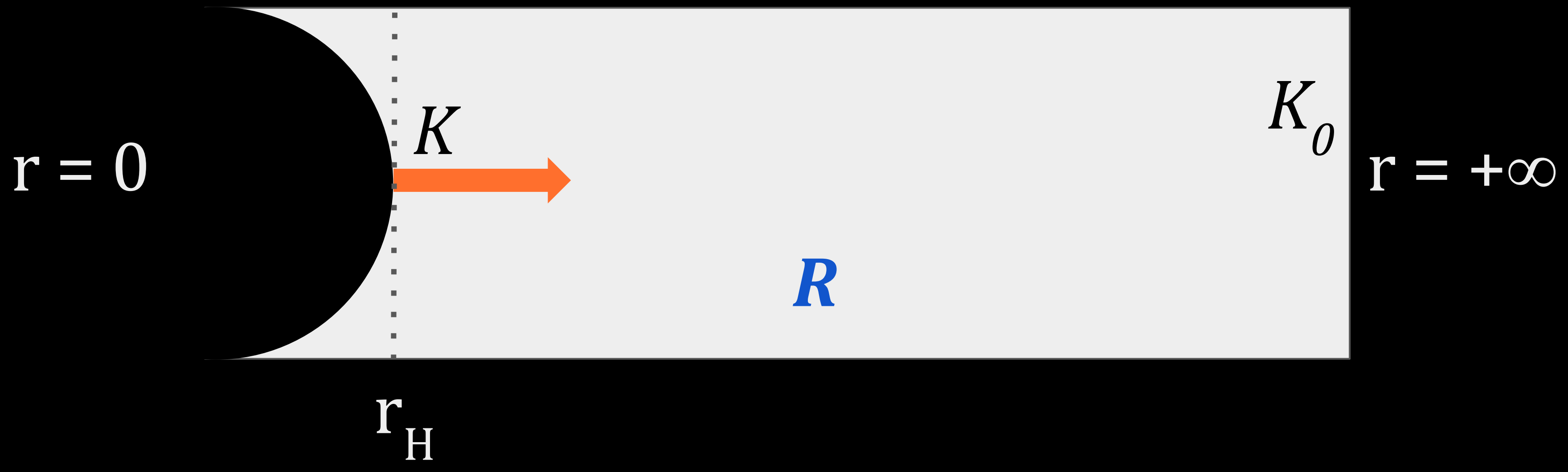
$$r \to \tilde{r} = r_H + \lambda(r - r_H)$$



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Derrick's argument: Convenient gauge

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Derrick's argument: Convenient gauge

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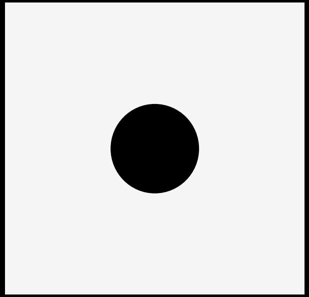

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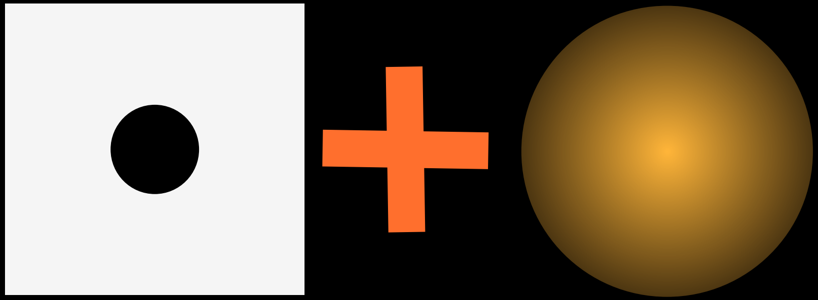

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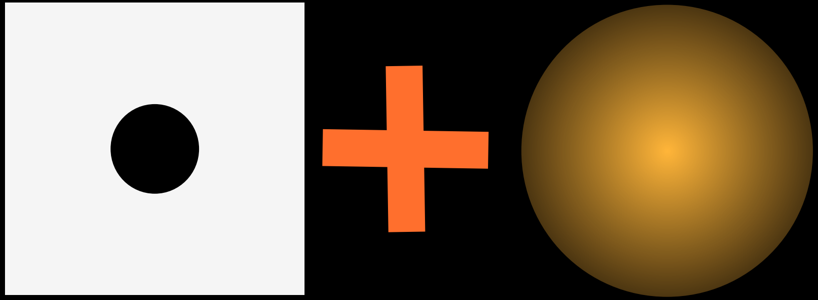

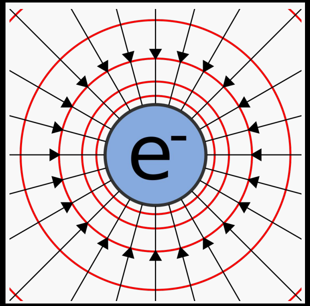

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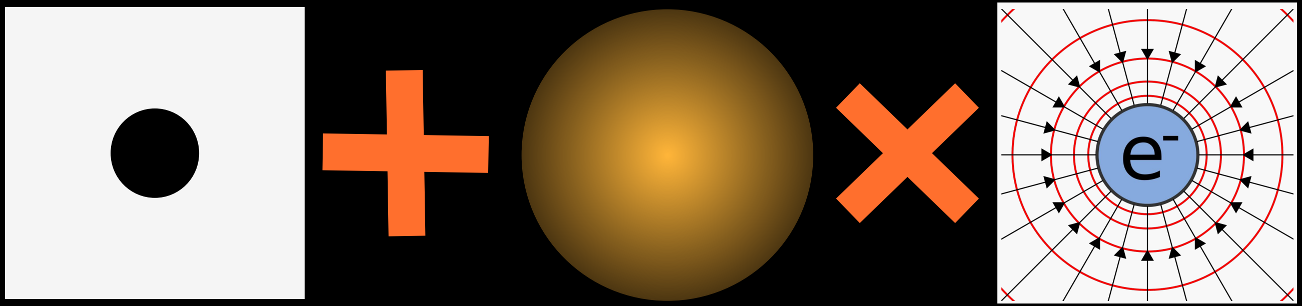

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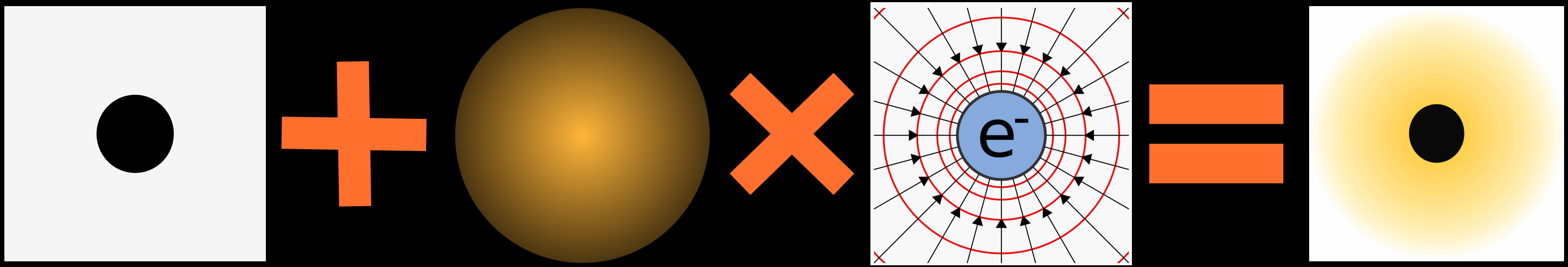

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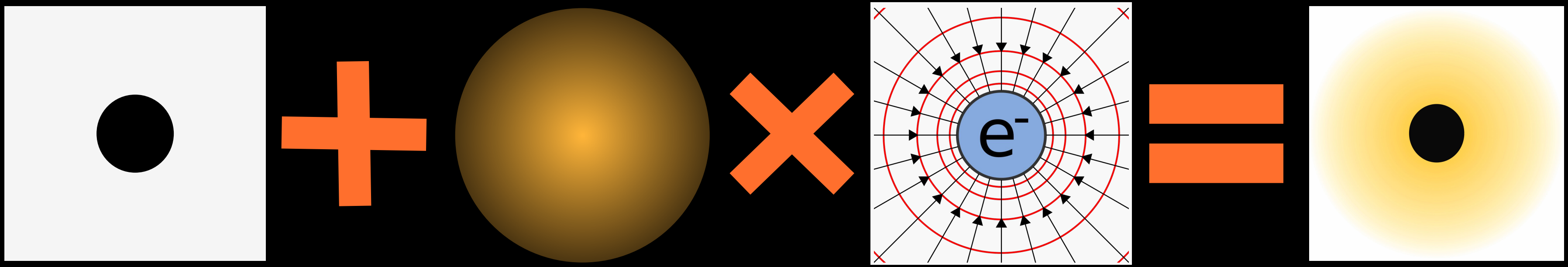

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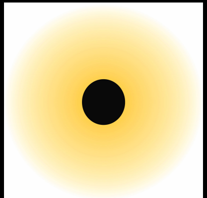

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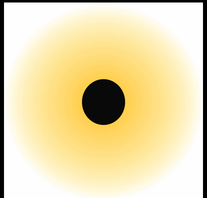

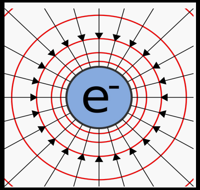

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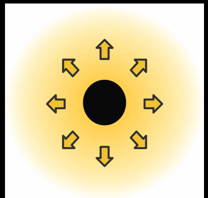

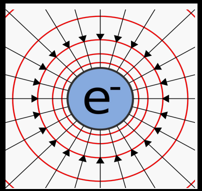

Second dish

Axial Symmetry

PRD 106 (2022)

Derrick’s argument: Q-balls

- The action of a complex scalar field

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$$\mathcal{S}^{\Phi} = \int d^4x\sqrt{-g}\Big[-\frac{1}{2}g^{\mu\nu}\left(\Phi_{,\mu}\Phi^*_{,\nu} + \Phi^*_{,\mu}\Phi_{,\nu}\right) - U(|\Phi|)\Big]$$

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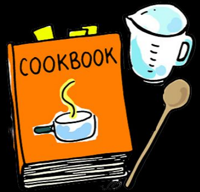

$$\int_0^{\pi} d\theta \int_0^{+\infty} dr\, r^2 \sin\theta \left[-3\omega^2 \phi^2 + \phi'^2 + \frac{\phi_{,\theta}^2}{r^2} + \frac{m^2 \phi^2}{r^2 \sin^2\theta} + 3U \right] = 0$$

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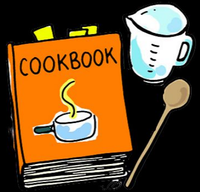

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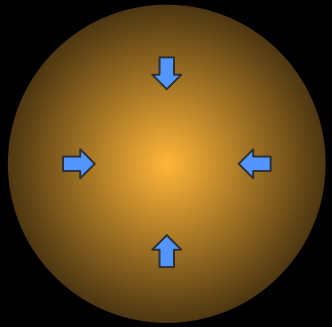

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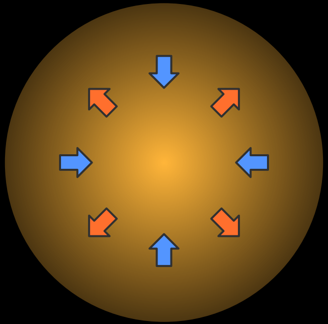

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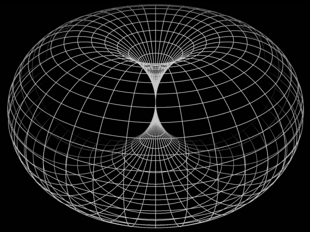

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$$S_{grav} = S_{EH} + S_{GHY}$$

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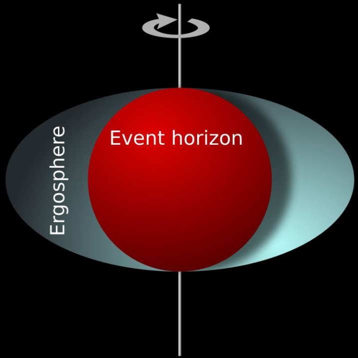

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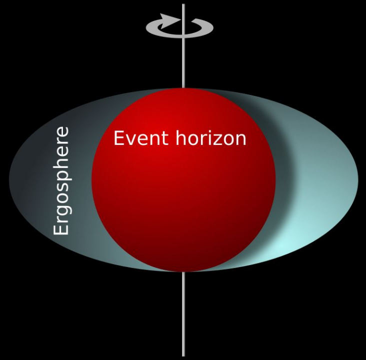

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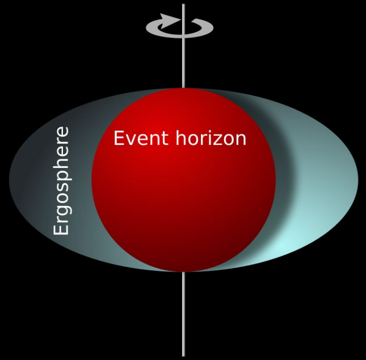


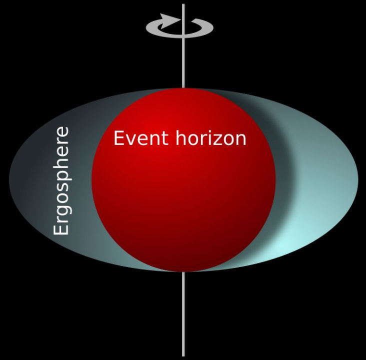
Derrick's argument: Kerr

- Let us pick the gravitational action again:

$$\mathcal{S}_{grav} = \mathcal{S}_{EH} + \mathcal{S}_{GHY} \quad \Longrightarrow \quad \int_0^{\pi} d\theta \int_{r_H}^{+\infty} dr\, I_R = \boxed{I_{GHY}}$$

$$ds^2 = -e^{2F_0} N dt^2 + e^{2F_1}\left(\frac{dr^2}{N} + r^2 d\theta^2\right) + e^{2F_2} r^2 \sin^2\theta (d\varphi - W dt)^2$$

$$N = 1 - \frac{r_H}{r}$$



Black Holes: Kerr

- The Gibbons-Hawking-York comes as

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$$\sqrt{-\gamma} = e^{F_0+F_1+F_2}\sqrt{N}\, r^2 \sin\theta\,,$$

$$K = \frac{e^{-F_1}}{r\sqrt{N}}\left[\frac{rN'}{2} + 2N + Nr\left(F_0' + F_1' + F_2'\right)\right],$$

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Black Holes: Hairy Kerr

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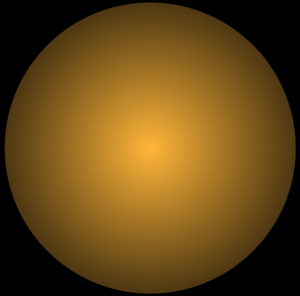

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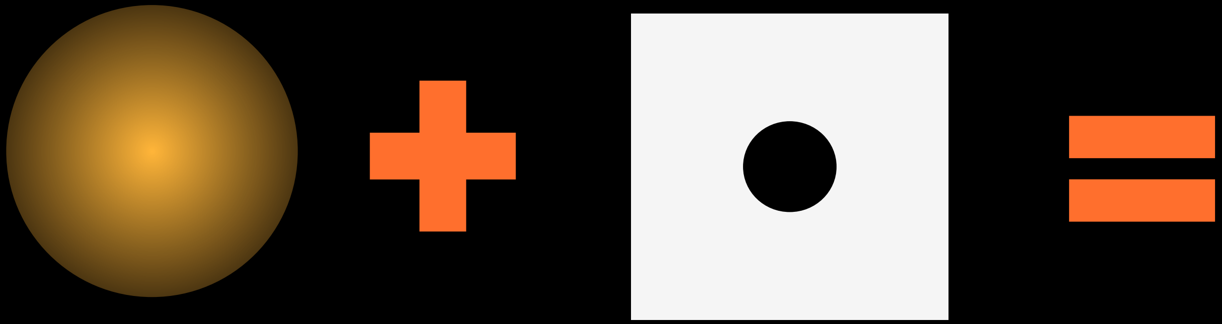

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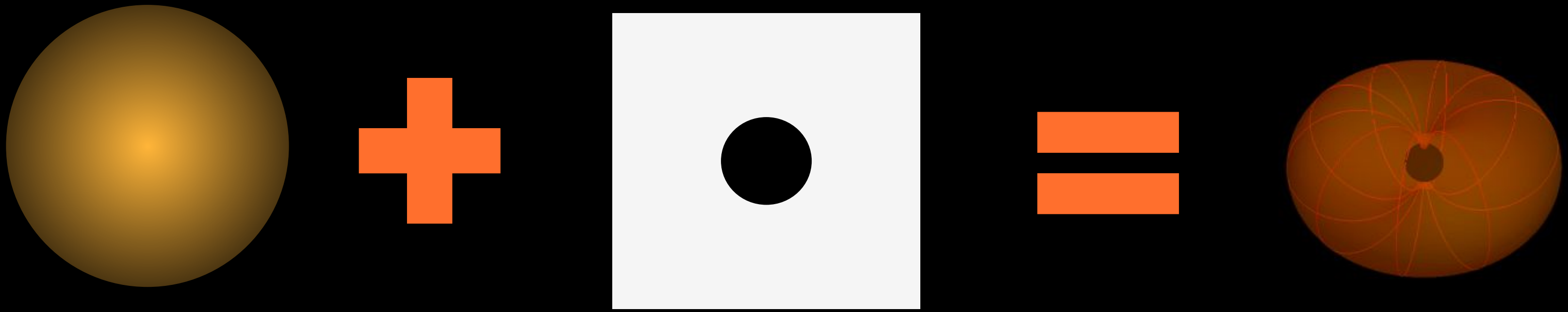

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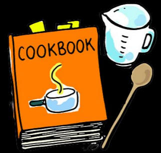

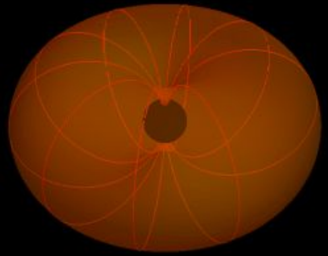

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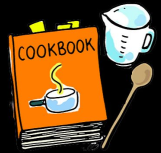

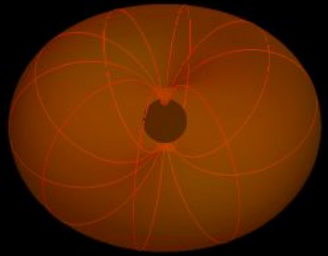

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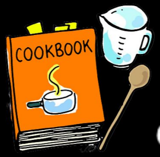
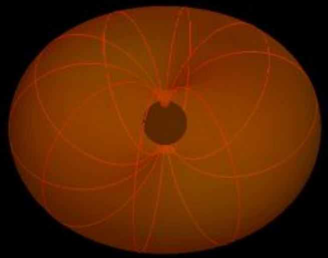


$$e^{F_0+F_2}\sin\theta\left[r^2 N^2 \phi_{,r}^2 + \phi_{,\theta}^2 + e^{2(F_1-F2)}\frac{m^2\phi^2}{\sin^2\theta}\right.$$
$$\left.+\, e^{2F_1} r^2\left\{\left(1-\frac{2r_H}{3r}\right)U - e^{-2F_0}\left(\omega - mW\right)^2\phi^2\right\}\right]$$



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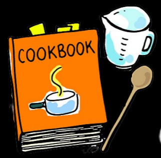

$$e^{F_0+F_2}\sin\theta\Big[r^2N^2\phi^2_{,r} + \phi^2_{,\theta} + e^{2(F_1-F2)}\frac{m^2\phi^2}{\sin^2\theta} + e^{2F_1}r^2\left\{\left(1-\frac{2r_H}{3r}\right)U - e^{-2F_0}(\omega - mW)^2\phi^2\right\}\Big]$$

$$I_R = -\frac{1}{2}\sin\theta e^{F_2-F_0}\Big(-2r_He^{2F_0}\left(2\left((r-r_H)\left(F_0''+F_1''+F_2''\right)+F_2'+(r-r_H)F_2'^2\right)+F_0'\left(2(r-r_H)F_2'+3\right)+2(r-r_H)F_0'^2+F_1'\right)$$
$$+2e^{2F_0}\Big(2\left(r(r-r_H)\left(F_0''+F_1''+F_2''\right)+r(r-r_H)F_2'^2+(3r-2r_H)F_2\right)-F_0'\left(-2r(r-r_H)F_2'-4r+r_H\right)+2\hat{\hat{F}}_0+2r(r-r_H)F_0'^2+2\hat{\hat{F}}_0$$
$$+(2r-r_H)F_1'+2\hat{\hat{F}}_2+2\hat{F}_2\left(\hat{F}_2+2\cot\theta\right)\Big)+4e^{2F_0}\hat{F}_0\left(\hat{F}_2+\cot\theta\right)+4e^{2F_0}\hat{F}_0^{\,2}+r^3r_H\sin^2\theta e^{2F_2}W'^2-3r^2\sin^2\theta e^{2F_2}\left(r(r-r_H)W'^2+\hat{W}^2\right)\Big)$$

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$$I_R = -\tfrac{1}{2}\sin \ldots$$

$$\theta e^{F_1-F_0}\left(-2r_H e^{2F_0}\left(2\left((r-r_H)\right(\ldots\right)$$

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$$+\,(2r-r_H)F_1'+2\hat{\hat{F}}_2+2\hat{F}_2\left(\hat{F}_2\ldots \left(r(r-r_H)W'^2+\hat{W}^2\right)\right)$$

Second dish

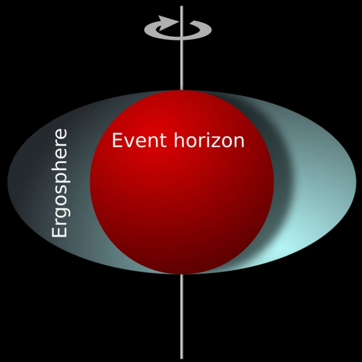
Convenient metric

PLB 837 (2023)

Derrick's argument: Black holes

- Let us introduce the a new metric ansatz:

$$ds^2 = -F_0 dt^2 + H F_1 dr^2 + P(r) F_1^2 d\theta^2 + F_2 (d\varphi - F_W dt)^2$$



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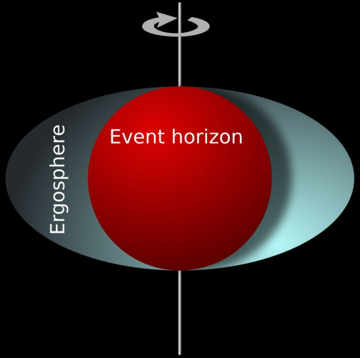

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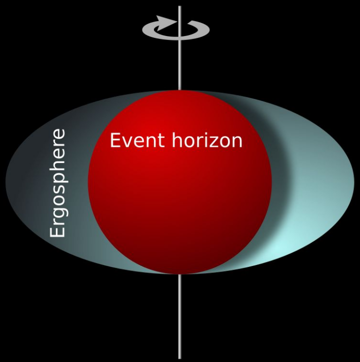

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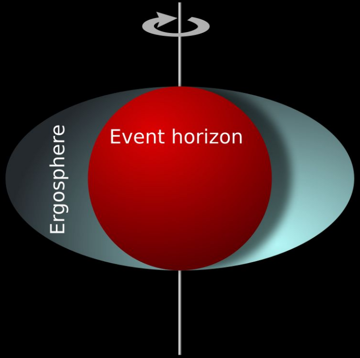

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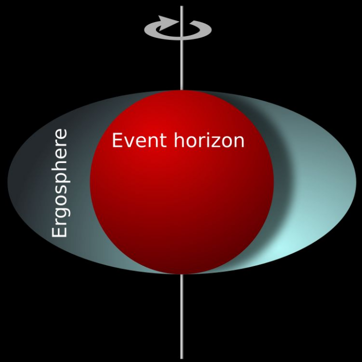

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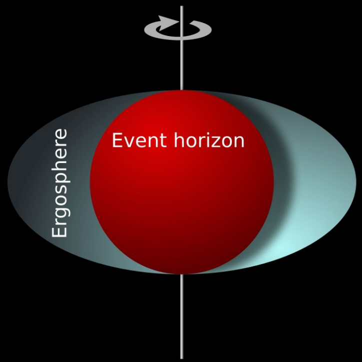

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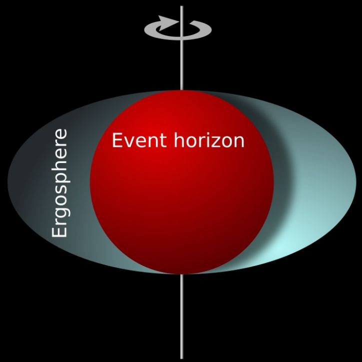
$$=> \; P(r) = (r - r_H)$$



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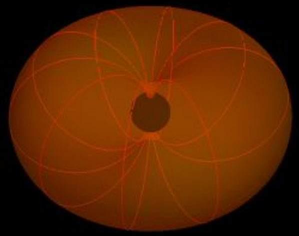


Dessert

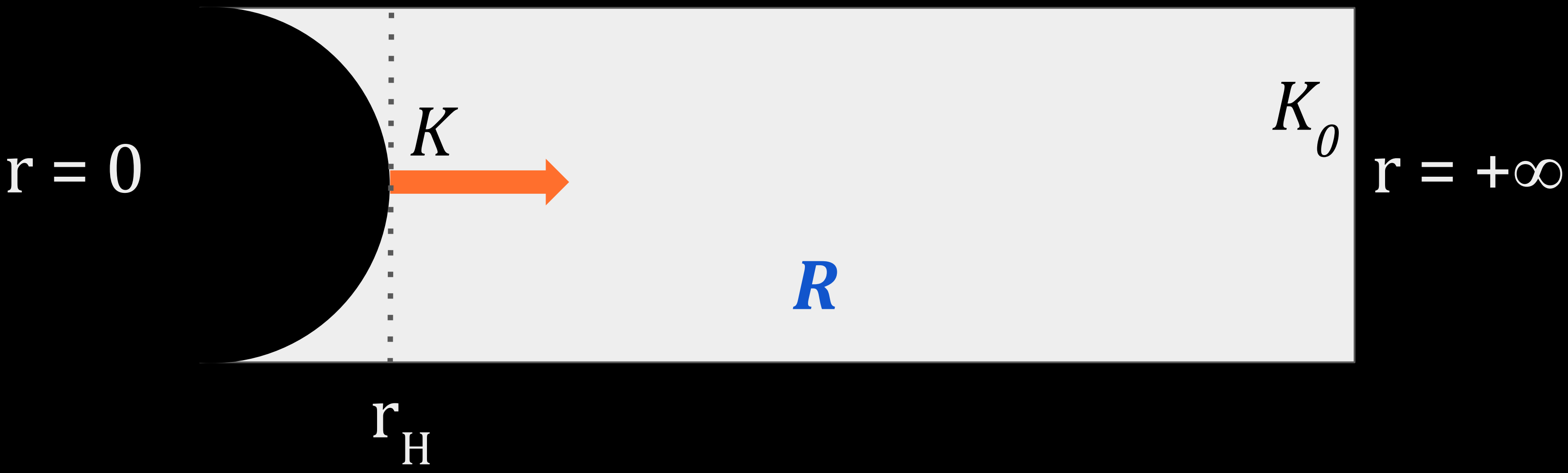
de Sitter

PRD (2025)

Derrick's argument: Black hole

- In the presence of an horizon

$$r \to \tilde{r} = r_H + \lambda(r - r_H)$$



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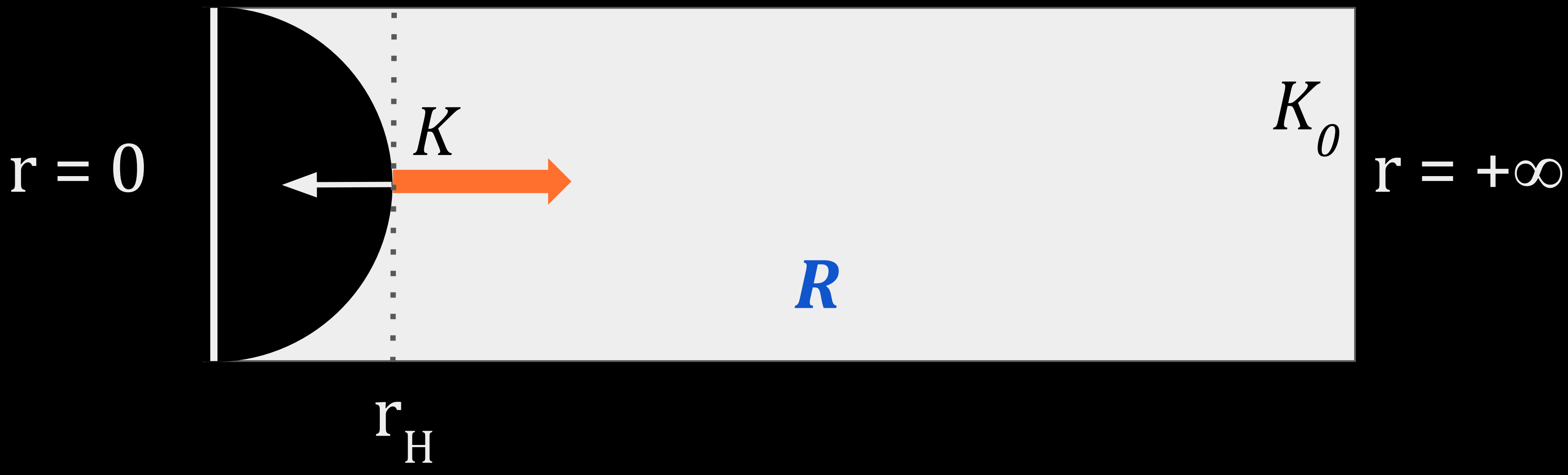

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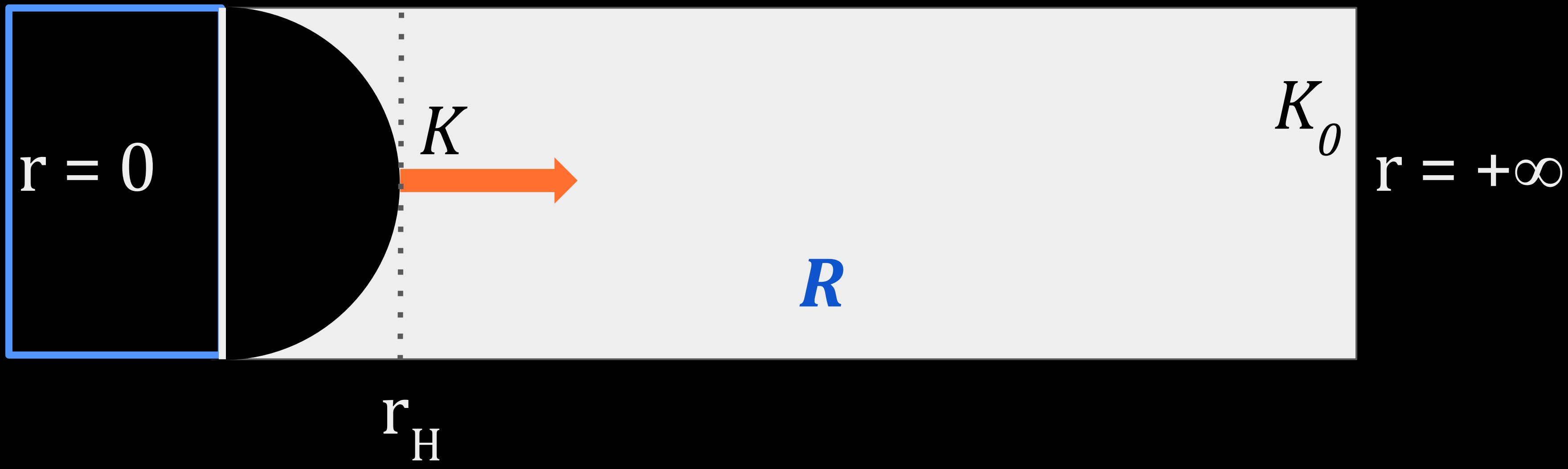

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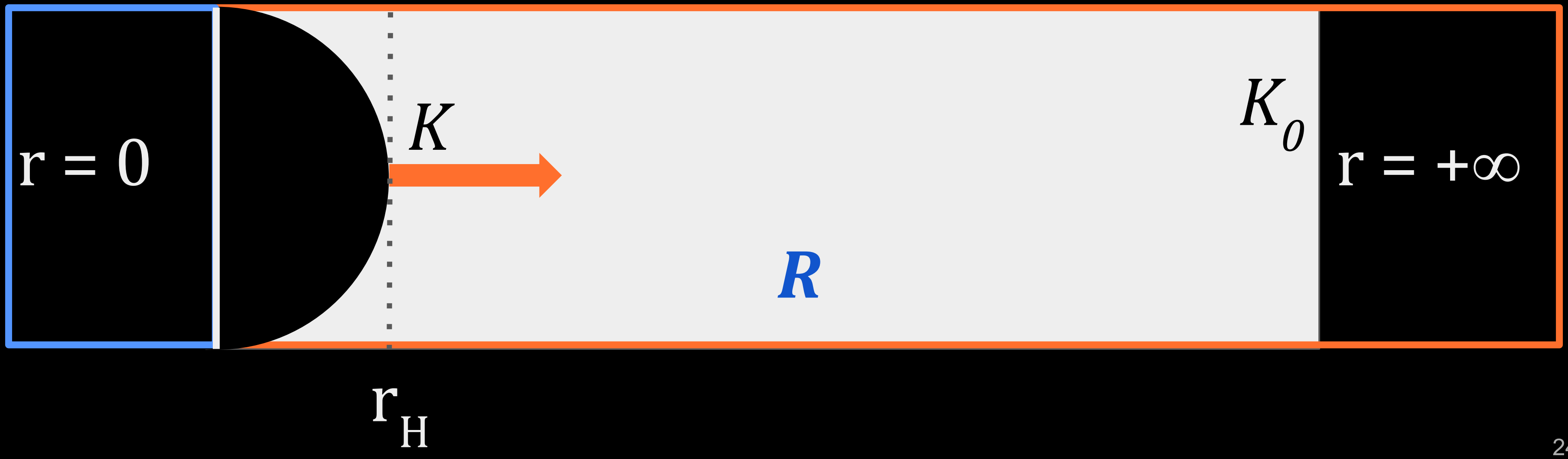
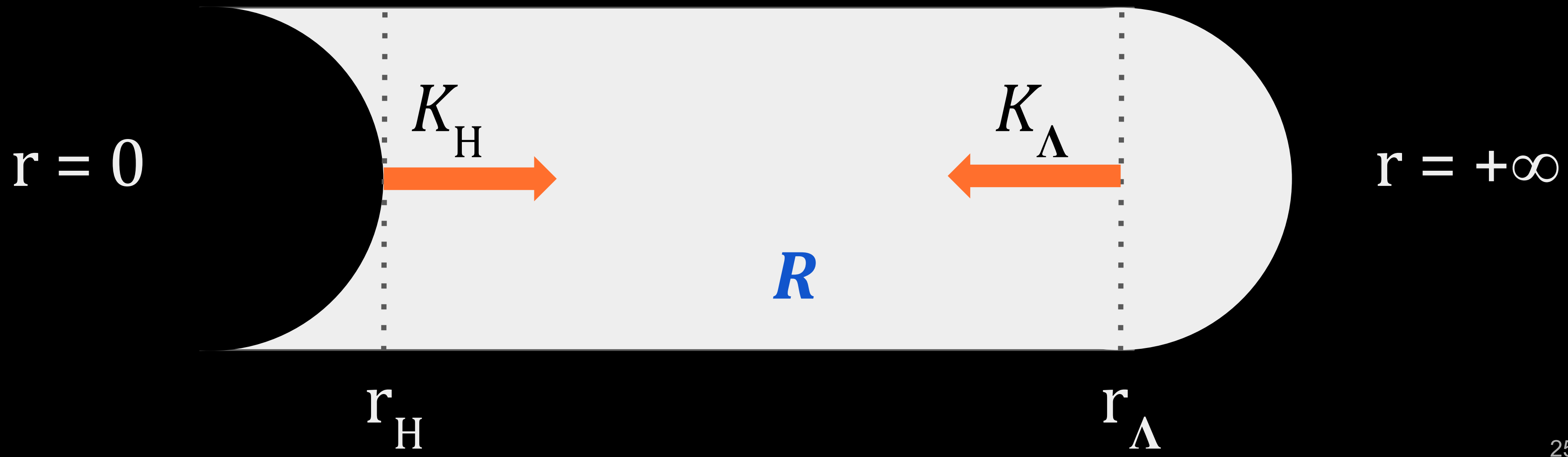


Derrick's argument: de Sitter

- In the presence of a cosmological horizon



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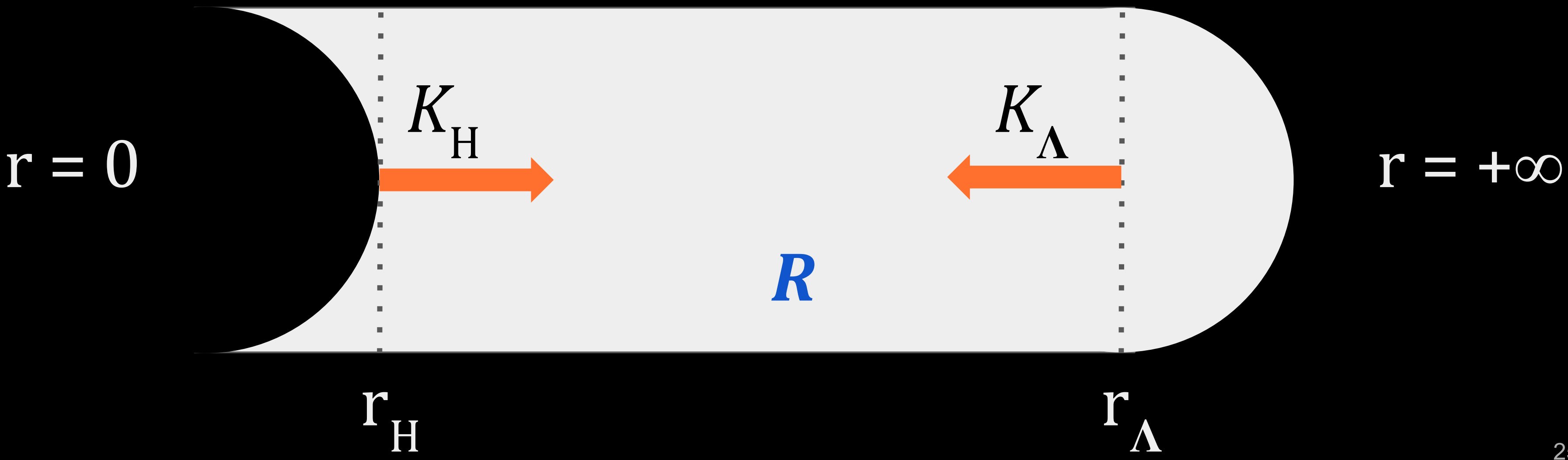

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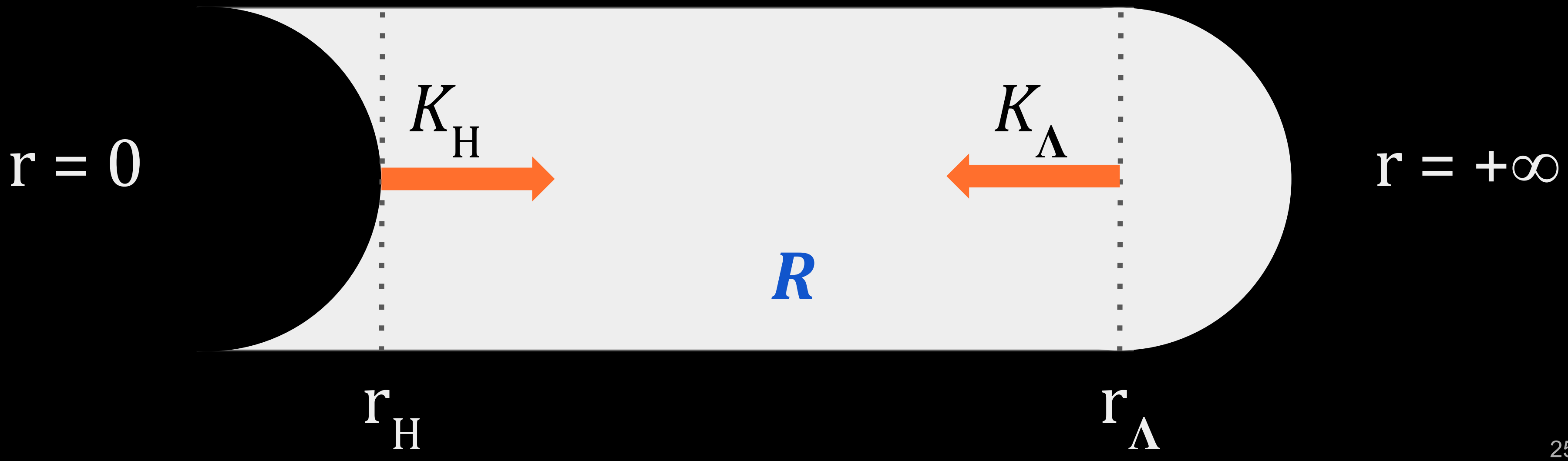
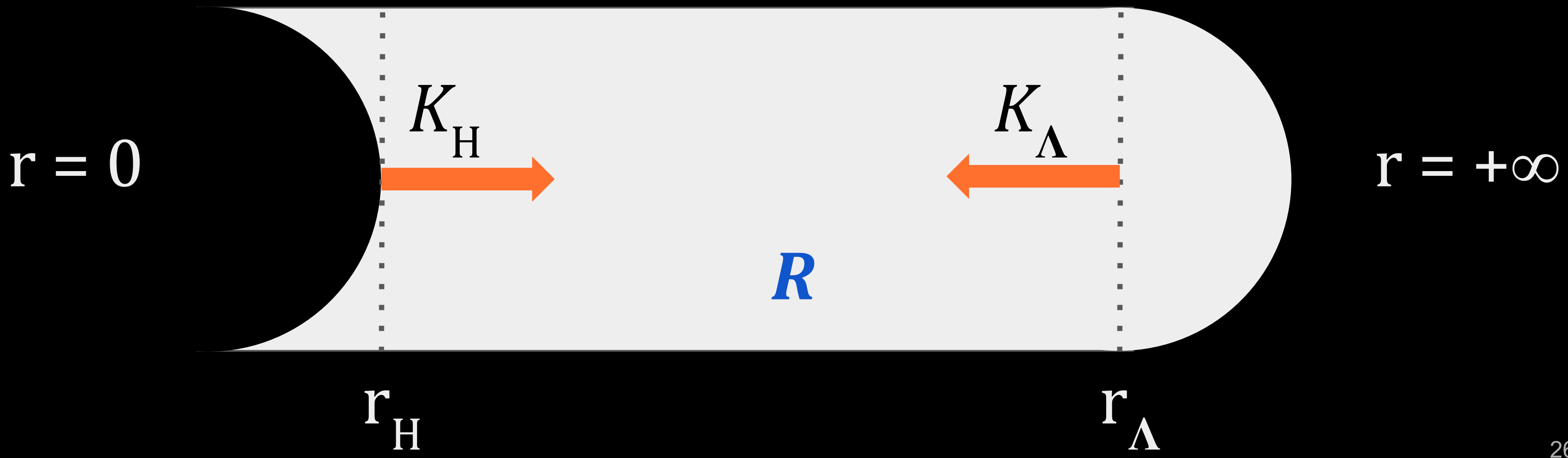


Derrick's argument: de Sitter

- In the presence of a cosmological horizon

$$r \to \frac{\tilde{r} - r_H}{r_\Lambda - r} r_\Lambda$$



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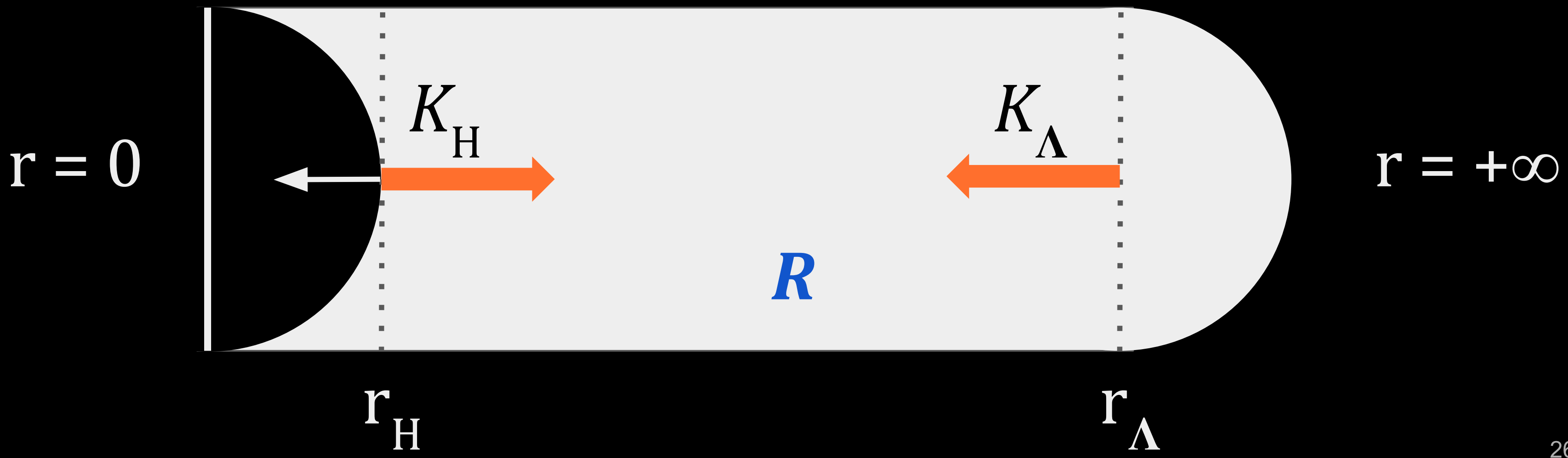

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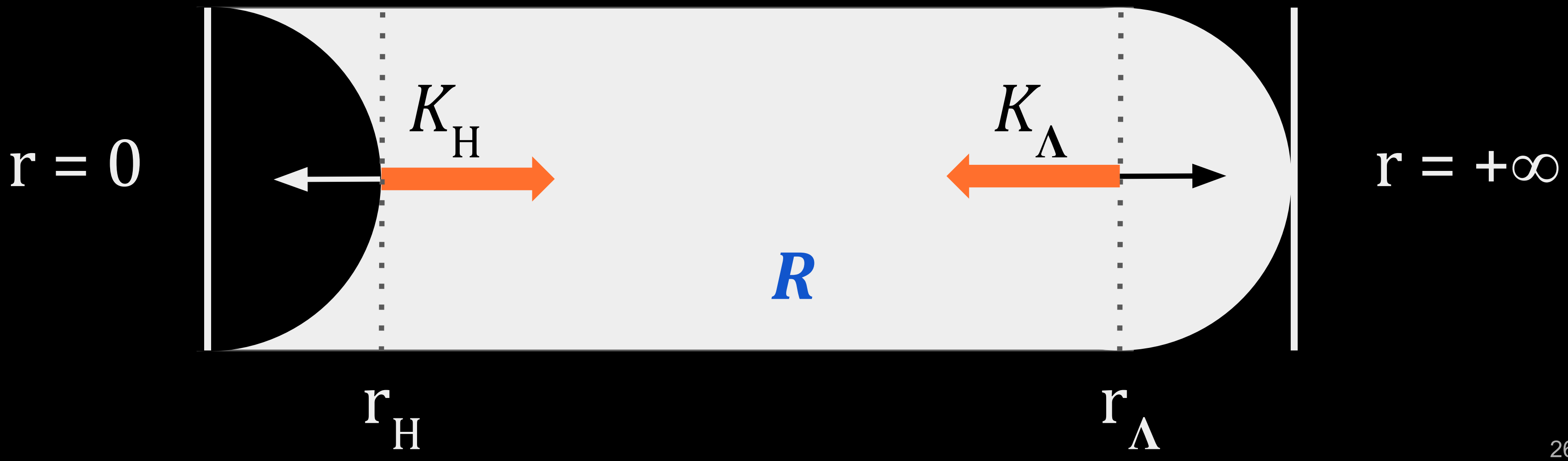

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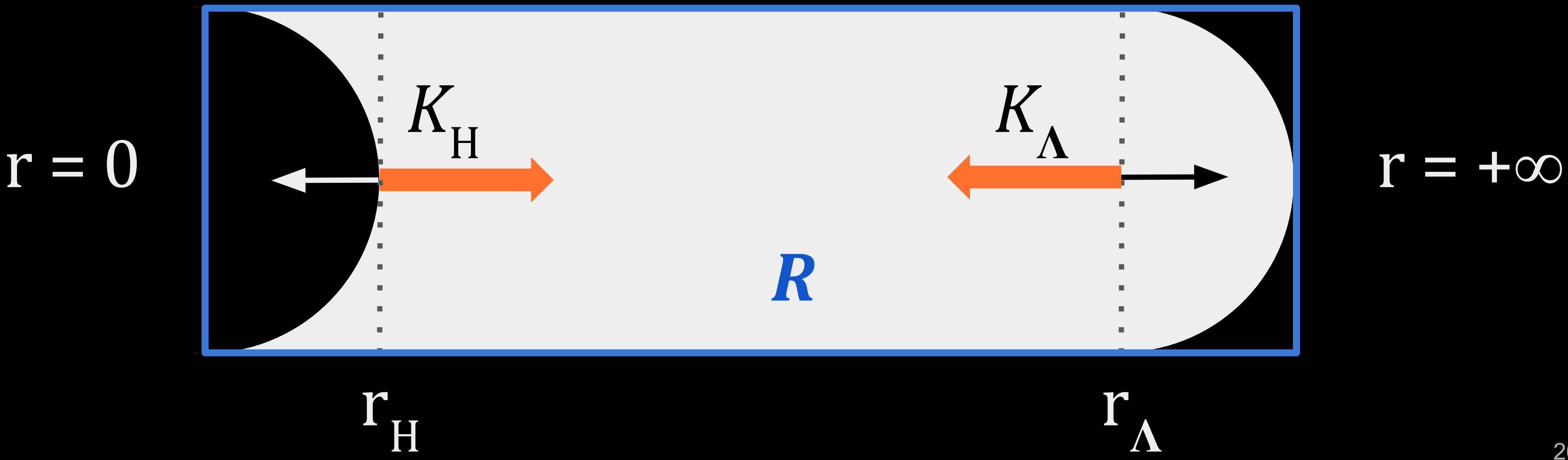
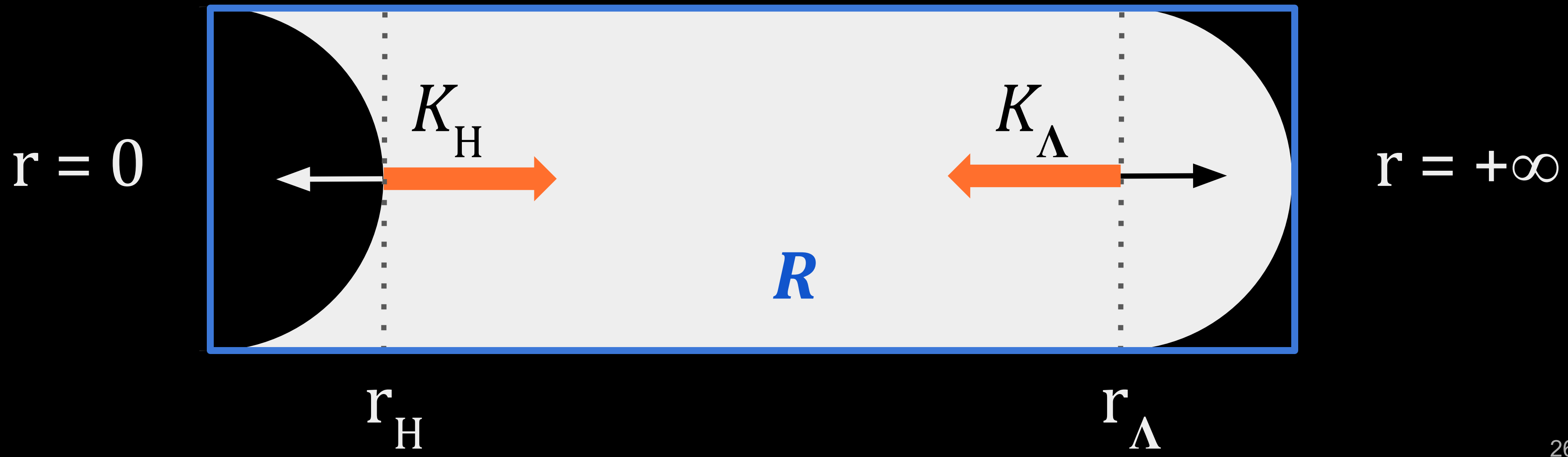


Derrick's argument: Schwarzschild-de Sitter

- In the presence of a cosmological horizon

$$r \to \frac{\tilde{r} - r_H}{r_\Lambda - r} r_\Lambda$$



Derrick's argument: Gravitational action

- The gravitational action

$$ds^2 = \boxed{-\sigma(r)^2}\boxed{N(r)dt^2} + \frac{dr^2}{N(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

1 **?**

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Derrick's argument: Gravitational action

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$$S_{grav} = S_{EH} + S_{\Lambda} + S_{GHY}$$

$$\int_{r_H}^{r_\Lambda} dr[I_R - I_\Lambda] = I_{GHY} = 0$$

Conclusion

Summary

- We presented a generic recipe to compute virial identities in field theory
- The GHY term is required due to the presence of second-order derivatives of the metric
- One noticed that, for a generic metric, relations are too complex
- There is a special "gauge" choice that trivializes the gravitational contribution
- In the presence of a second boundary, an alternative radial transformation is necessary.

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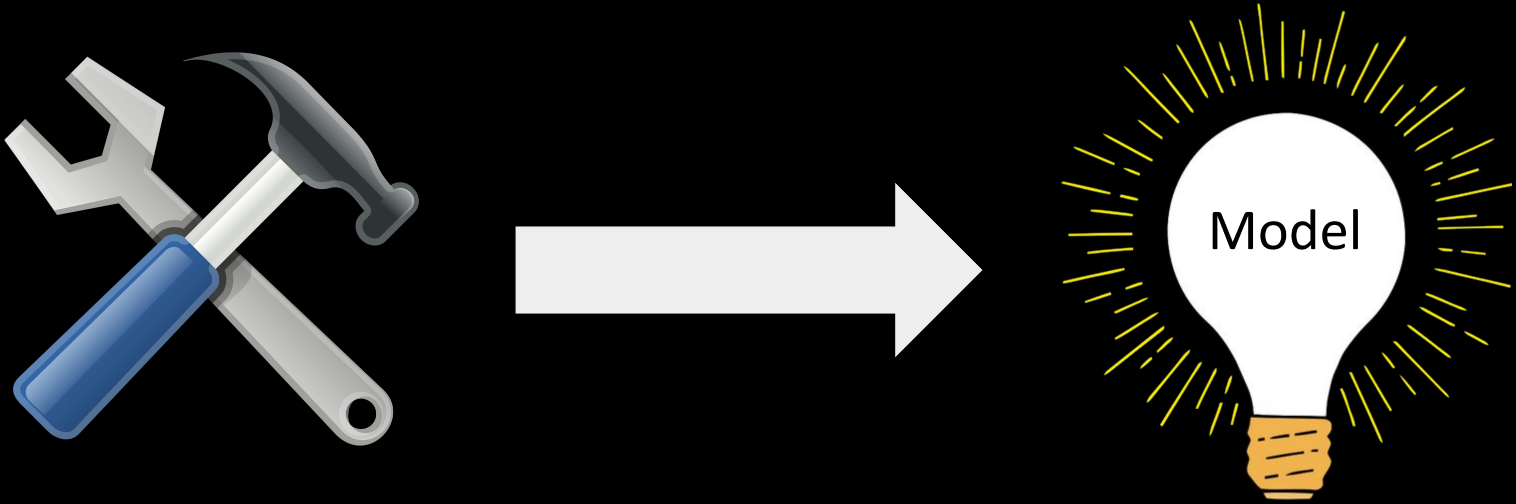
Conclusion

Virial identities are a helpful tool that can be used to have a better insight into the models



Virial **Insight**

Thanks

Děkuju!

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IN COSMOLOGY,
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PHYSICS

Virial identities across the spacetime

2109.05027
2206.02813
2207.12451
2406.00112

Thank you!

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