# Black holes with primary scalar hair

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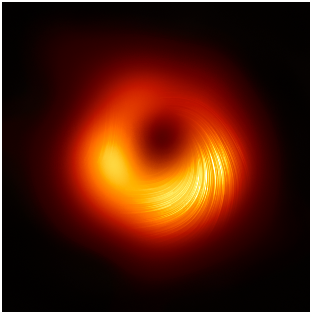
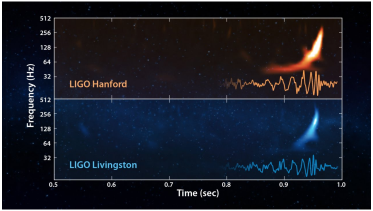


## Event Horizon Telescope (EHT)

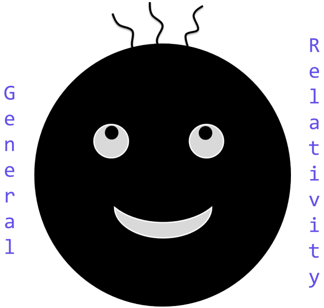
- **Goal**: Direct observation of black holes' photon ring.
- **Major Achievements**: First-ever image of a black hole (M87*) in 2019.
- **Nobel Prize in Physics 2020:** Roger Penrose, Reinhard Genzel, and Andrea Ghez.




## LIGO/Virgo/KAGRA

- **Goal**: Detecting gravitational waves from compact-object mergers.
- **Major Achievements**: First detection of black hole–black hole merger (GW150914, 2015).
- **Nobel Prize in Physics 2017:** Rainer Weiss, Barry C. Barish, and Kip S. Thorne.

# Black holes have no hair!




## GR solutions are special

In General Relativity, every black hole is characterized by only three observable quantities/“hairs”:

- Mass ($M$)
- Angular Momentum/Spin ($\mathcal{J}$)
- Electric charge ($Q_e$)

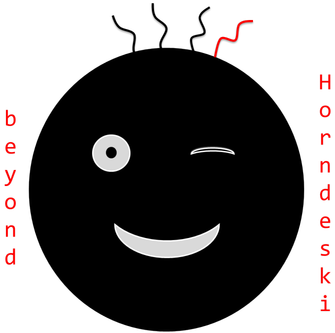
Two black holes with the same values for these parameters are completely indistinguishable.

## No(-scalar)-hair theorems

- J. D. Bekenstein, 1972 & 1995
- J. D. Bekenstein [9605059/gr-qc] (short review)
- C. Herdeiro, E. Radu [1504.08209/gr-qc] (review)

More often than not, these theorems can be evaded in Einstein-scalar-Gauss-Bonnet gravity or Horndeski and beyond Horndeski theories.

# Black holes have ~~no~~ hair!




In gravitational theories beyond GR, black holes might acquire additional properties, depending on the theoretical framework.

## Scalar-tensor (ST) theories

For static and asymptotically flat black-hole solutions in ST theories:

- Secondary hair: $g_{\mu\nu} = g_{\mu\nu}(M; x^{\lambda})$, $\phi = \phi(M; x^{\lambda})$ (no additional information)
- Primary hair: $g_{\mu\nu} = g_{\mu\nu}(M, \boldsymbol{q}; x^{\lambda})$,
  $\phi = \phi(M, \boldsymbol{q}; x^{\lambda})$
  (additional information linked to the existence of the scalar field)

In [2310.11919/gr-qc] we presented *the first solution of a black hole with primary scalar hair in a single field scalar-tensor theory (beyond Horndeski gravity).*

In [2312.17198/gr-qc], *we generalized the method and obtained a class of different solutions with primary scalar hair.*

# Bocharova-Bronnikov-Melnikov-Bekenstein solution (1970s)

## A scalar-tensor (ST) theory with a conformally coupled scalar field

$$S[g_{\mu\nu},\phi] = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{16\pi} - \frac{1}{12}\phi^2 R - \frac{1}{2}\partial_\lambda\phi\,\partial^\lambda\phi\right)$$

Invariance of the EOM of $\phi$ under the conformal transformation:

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}\,, \quad \tilde{\phi} = \Omega^{-1}\phi$$

## The BBMB solution

- There exist **static** and **spherically symmetric** black-hole solutions

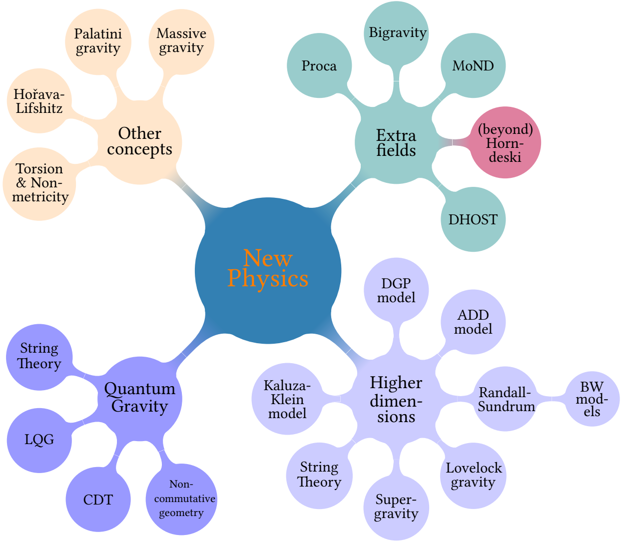
$$\mathrm{d}s^2 = -\left(1-\frac{M}{r}\right)^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1-\frac{M}{r}\right)^2} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2)\,.$$

- The profile of the scalar field is non-trivial

$$\phi = \sqrt{\frac{3}{4\pi}}\frac{M}{r-M}\,.$$

- The black holes possess **secondary hair**.

# Scalar-tensor theories, but why?



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- They are the **simplest modifications of gravity** with a single scalar degree of freedom (e.g. Brans-Dicke, Horndeski, beyond Horndeski, DHOST) .
- They constitute limits of more complex fundamental theories:
  - Lovelock $\xrightarrow{\text{Kaluza-Klein reduction}}$ Horndeski
- String Theory predicts that the actual theory of gravity is a scalar–tensor theory. The spin-2 graviton is accompanied by a spin-0 partner, the dilaton.

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Scalar-tensor theories are the simplest well-motivated departures from GR

# Horndeski and beyond Horndeski theories

## Horndeski gravity (1974)

$$S_H[g_{\mu\nu},\phi] = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}\left\{\mathcal{L}_2^H + \mathcal{L}_3^H + \mathcal{L}_4^H + \mathcal{L}_5^H\right\},$$

$$\mathcal{L}_2^H = G_2(\phi,X)\,,\qquad \mathcal{L}_3^H = -G_3(\phi,X)\Box\phi\,,\qquad \mathcal{L}_4^H = G_4(\phi,X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right],$$

$$\mathcal{L}_5^H = G_5(\phi,X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right].$$

Here, $X = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi$ represents the kinetic term, $G_{iX} \equiv \mathrm{d}G_i/\mathrm{d}X$, while $(\nabla_\mu\nabla_\nu\phi)^2 \equiv (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)$ and $(\nabla_\mu\nabla_\nu\phi)^3 \equiv (\nabla_\mu\nabla_\nu\phi)(\nabla^\nu\nabla^\lambda\phi)(\nabla_\lambda\nabla^\mu\phi)$.

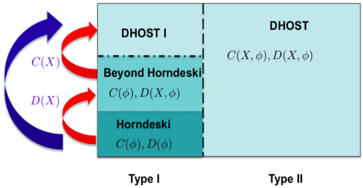


Figure stolen from David Langlois [1811.06271/gr-qc]

## General disformal transformation

$$\tilde{g}_{\mu\nu} = C(X,\phi)g_{\mu\nu} + D(X,\phi)\partial_\mu\phi\partial_\nu\phi$$

## Horndeski → Beyond Horndeski

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + D(X)\partial_\mu\phi\partial_\nu\phi$$

# Black holes with primary scalar hair

A. Bakopoulos, C. Charmousis, N. Lecoeur, P. Kanti, T.N., [2310.11919/gr-qc]
A. Bakopoulos, N. Chatzifotis, T.N., [2312.17198/gr-qc]

- Shift symmetric ($\phi \to \phi + c$) and $Z_2$ symmetric ($\phi \to -\phi$) beyond Horndeski theory:

$$S_{bH}\left[g_{\mu\nu},\phi\right] = \frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\Big\{G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right] + F_4(X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma}{}_{\sigma}\phi_{,\mu}\phi_{,\alpha}\phi_{;\nu\beta}\phi_{;\rho\gamma}\Big\},$$

$$G_{iX} \equiv \frac{\mathrm{d}G_i}{\mathrm{d}X}, \quad \phi_{,\mu} \equiv \partial_\mu\phi, \quad \phi_{;\mu\nu} \equiv \nabla_\mu\partial_\nu\phi, \quad X \equiv -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi.$$

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- We seek static, spherically symmetric and asymptotically flat solutions

$$\mathrm{d}s^2 = -h(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2)\,, \qquad \phi(t,r) = qt + \psi(r)\,.$$

# Black holes with primary scalar hair

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- The **internal shift symmetry of the theory** $\Rightarrow$ **a Noether current**

$$\left\{\mathbf{J} = \mathcal{J}_\mu dx^\mu, \quad \mathcal{J}^\mu = \frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta(\partial_\mu\phi)} = (\mathcal{J}^t, 0, 0, 0)\right\} \Rightarrow \boxed{Q_s = \int \star\mathbf{J} \propto q^k}$$

Using the auxiliary function $Z(X) \equiv 2XG_{4X} - G_4 + 4X^2 F_4$, the independent EOM are

$$\frac{f}{h} = \frac{\gamma^2}{Z^2}\,, \tag{1}$$

$$r^2 (G_2 Z)_X + 2(G_4 Z)_X \left(1 - \frac{q^2\gamma^2}{2Z^2 X}\right) = 0\,, \tag{2}$$

$$2\gamma^2 \left(hr - \frac{q^2 r}{2X}\right)' = -r^2 G_2 Z - 2G_4 Z \left(1 - \frac{q^2\gamma^2}{2Z^2 X}\right) + \frac{q^2\gamma^2 X' r}{Z X^2} \left(2XG_{4X} - G_4\right)\,. \tag{3}$$

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For **homogeneous solutions ($h = f$)**, the above system of equations is **integrable**.

- Eq. (1) results in $Z = \gamma$.
- Assuming that $G_4(X) = \frac{\lambda^2}{2}G_2(X) + \zeta$, eq. (2) yields

$$X = \frac{q^2}{2}\frac{1}{1+(r/\lambda)^2}\,.$$

- Eq. (3) now leads to

$$h(r) = 1 + \frac{C}{r} + \left(1 + \frac{\zeta}{\gamma}\right)\frac{r^2}{\lambda^2} + \frac{1}{\gamma}\frac{1}{r}\int r^2\left(G_2 - 2XG_{2X}\right)\mathrm{d}r\,.$$

Considering

$$G_2(X) = \sum_{n=0}^{\infty} c_{\frac{n}{s}} X^{\frac{n}{s}}\,, \quad s \in \mathbb{Z}^+\,, \quad [c_{\frac{n}{s}}] = [L]^{2\left(\frac{n}{s}-1\right)}$$

one obtains

$$h(r) = 1 + \frac{C}{r} + \left(1 + \frac{\zeta}{\gamma} + \frac{\lambda^2}{3\gamma} c_0\right)\frac{r^2}{\lambda^2} + \frac{r^2}{3\gamma}\sum_{n=1}^{\infty} c_{\frac{n}{s}}\left(1 - \frac{2n}{s}\right)\left(\frac{q^2/2}{1+(r/\lambda)^2}\right)^{\frac{n}{s}} {}_2F_1\left(\frac{n}{s}, 1; \frac{5}{2}; \frac{1}{1+\lambda^2/r^2}\right).$$

At $r \to +\infty$ one finds

$$h(r) = 1 + \frac{1}{r}\left[C + \frac{\lambda^3\sqrt{\pi}}{4\gamma}\sum_{n=1}^{\infty} c_{\frac{n}{s}}\left(1 - \frac{2n}{s}\right)\left(\frac{q^2}{2}\right)^{\frac{n}{s}} \frac{\Gamma\left(\frac{n}{s} - \frac{3}{2}\right)}{\Gamma\left(\frac{n}{s}\right)}\right] + \left(1 + \frac{\zeta}{\gamma} + \frac{\lambda^2}{3\gamma} c_0\right)\frac{r^2}{\lambda^2}$$
$$+ \frac{2\beta}{3\gamma}\sum_{n=1}^{\infty} c_{\frac{n}{s}}\left(\frac{q^2}{2}\right)^{\frac{n}{s}}\left(\frac{\lambda}{r}\right)^{\frac{2n}{s}}\left[\left(\frac{1-\frac{2n}{s}}{1-\frac{2n}{3s}}\right)\frac{r^2}{\lambda^2} - \frac{3n}{s} + \mathcal{O}\left(\frac{1}{r^2}\right)\right].$$

For asymptotically flat solutions, it is necessary to have $\zeta = -\gamma = 1$, $c_0 = 0$, and $\frac{n}{s} > \frac{3}{2}$.

## 1st Solution

$$G_2(X) = \sum_{n=0}^{\infty} c_{\frac{n}{s}} X^{\frac{n}{s}} \,, \quad s \in \mathbb{Z}^+$$

$$c_{\frac{n}{s}} = 0, \quad \forall \frac{n}{s} \neq 2$$

$$c_2 = -\frac{8\eta}{3\lambda^2}$$

- Model functions of the theory:

$$G_2(X) = -\frac{8\eta}{3\lambda^2}X^2 \quad G_4(X) = 1 - \frac{4\eta}{3}X^2\,, \quad F_4(X) = \eta\,,$$

$\eta$ and $\lambda$ coupling constants, with dimensions $(\text{length})^4$ and $(\text{length})$, respectively.

- Spacetime geometry:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\,,$$

$$f(r) = 1 - \frac{2M}{r} + \eta q^4\left[\frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda} + \frac{1}{1+(r/\lambda)^2}\right]\,.$$

- Scalar field, kinetic term, and scalar charge/hair:

$$\phi(t,r) = qt + \psi(r)\,, \quad X = \frac{q^2/2}{1+(r/\lambda)^2}\,, \quad [\psi'(r)]^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right]\,,$$

$$J^\mu = \left(-\frac{2q}{1+(r/\lambda)^2}G_{2X}, 0, 0, 0\right)\,, \quad Q_s = \int \star\mathbf{J} = \frac{16\pi^2}{3}\eta\lambda q^3\,.$$

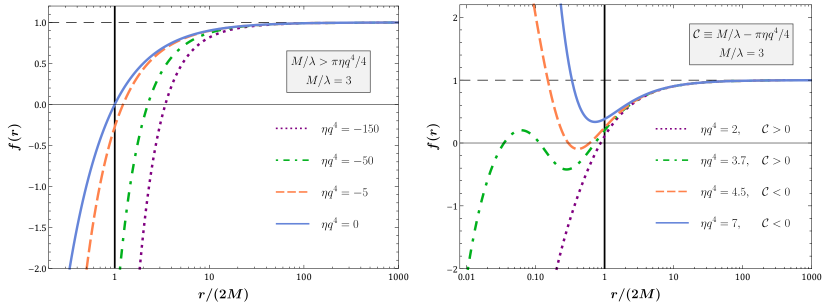
- The solution has **two independent free parameters**: $M$ (ADM mass), $q$ (primary scalar hair).
For $q = 0 \to \{\text{GR limit} + \text{Schwarzschild solution}\}$.

Asymptotically the solution behaves like RN but with the scalar playing the role of electric charge

$$f(r \to +\infty) = 1 - \frac{2M}{r} + 2\eta q^4 \frac{\lambda^2}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right),$$

while close to the singularity we have

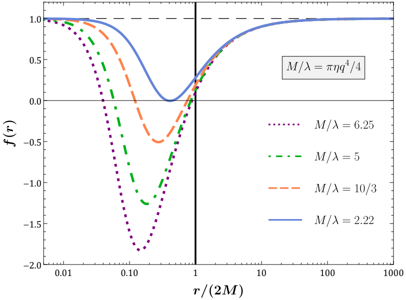
$$f(r \to 0) = 1 - \frac{2M - \pi\eta q^4\lambda/2}{r} - \frac{2\eta q^4 r^2}{3\lambda^2} + \mathcal{O}(r^4)\,.$$

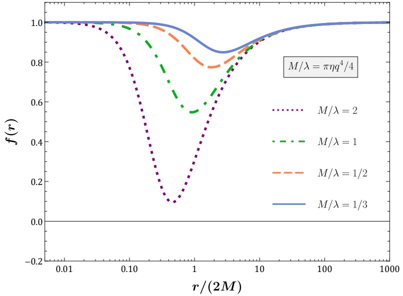



**Left:** $\eta < 0$, single-horizon BH more sparse than Schwarzschild. **Right:** $\eta > 0$, multiple-horizon BH more compact than Schwarzschild or naked singularities.

For $M = \pi\eta q^4\lambda/4$, the central singularity disappears altogether and all curvature invariants become infinitely regular:

$$f(r) = 1 - \frac{4M}{\pi\lambda}\left[\frac{\arctan(r/\lambda)}{r/\lambda} - \frac{1}{1+(r/\lambda)^2}\right].$$




**Left:** Regular BH solutions. **Right:** Solitonic solutions.

## 2nd Solution

$$G_2(X) = \sum_{n=0}^{\infty} c_{\frac{n}{s}} X^{\frac{n}{s}}\,, \quad s \in \mathbb{Z}^+$$

$$c_{\frac{n}{s}} = 0, \quad \forall \frac{n}{s} \neq \frac{5}{2}$$

$$c_{\frac{5}{2}} = \frac{2\eta}{\lambda^2}$$

- Model functions of the theory:

$$G_2(X) = \frac{2\eta}{\lambda^2} X^{5/2} \quad G_4(X) = 1 + \eta X^{5/2}\,, \quad F_4(X) = -\eta\sqrt{X}\,,$$

$\eta$ and $\lambda$ coupling constants, with dimensions $(\text{length})^5$ and $(\text{length})$, respectively.

- Spacetime geometry:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\,,$$

$$f(r) = 1 - \frac{2\boldsymbol{M}}{r} - \frac{\sqrt{2}\,\eta\boldsymbol{q}^5}{3}\frac{\lambda}{r}\left(1 - \frac{r^3}{(r^2+\lambda^2)^{3/2}}\right).$$

- Scalar field, kinetic term, and scalar charge/hair:

$$\phi(t,r) = \boldsymbol{q}t + \psi(r)\,, \quad X = \frac{q^2/2}{1+(r/\lambda)^2}\,, \quad [\psi'(r)]^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right]\,,$$

$$\mathcal{J}^\mu = \left(-\frac{2q}{1+(r/\lambda)^2}G_{2X}, 0, 0, 0\right)\,, \quad \boldsymbol{Q_s} = \int \star\mathbf{J} = -\frac{20\pi}{3\sqrt{2}}\eta\lambda\boldsymbol{q}^4\,.$$

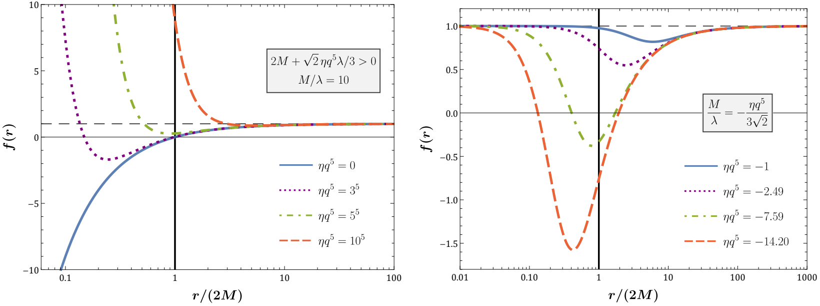
- The solution has **two independent free parameters**: $\boldsymbol{M}$ (ADM mass), $\boldsymbol{q}$ (primary scalar hair). For $q = 0 \rightarrow \{\text{GR limit} + \text{Schwarzschild solution}\}$.

Asymptotically the solution behaves like the Schwarzschild solution but with a small correction from the scalar hair

$$f(r \to +\infty) = 1 - \frac{2M}{r} - \frac{\eta q^5}{\sqrt{2}} \frac{\lambda^3}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right),$$

while close to the singularity we have

$$f(r \to 0) = 1 - \frac{2M + \sqrt{2}\,\eta q^5 \lambda/3}{r} - \frac{\sqrt{2}\,\eta q^5}{3} \frac{r^2}{\lambda^2} + \mathcal{O}(r^4).$$




**Left:** $\eta > 0$, BH solutions with two horizons. **Right:** $\eta < 0$, Regular BH and solitonic solutions.

## Bardeen solution in beyond Horndeski

Especially in the regular case, where $\frac{M}{\lambda} = -\frac{\eta q^5}{3\sqrt{2}}$, the resulting solution is the Bardeen:

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2)\,,$$

$$f(r) = 1 - \frac{2\boldsymbol{M}r^2}{(r^2+\lambda^2)^{3/2}}\,,$$

***M*** is a free parameter.

## Bardeen from non-linear magnetic monopole (E. Ayon-Beato, A. Garcia **[0009077/gr-qc]**)

$$S = \frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}[R - 4\mathcal{L}(\mathcal{F})]\,, \quad \mathcal{L}(\mathcal{F}) = \frac{3M}{\lambda^3}\left(\frac{\sqrt{2\lambda^2\mathcal{F}}}{1+\sqrt{2\lambda^2\mathcal{F}}}\right)^{5/2}\,,$$

$$\mathcal{F} \equiv \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \quad F_{\mu\nu} = 2\delta^{\vartheta}_{[\mu}\delta^{\varphi}_{\nu]}\lambda\sin\vartheta\,.$$

In this case, the parameter ***M*** is a coupling constant and therefore not a free parameter.

**The beyond Horndeski gravity constitutes a more natural framework to describe the Bardeen solution.**

## Conclusions:

- We have demonstrated a **generic method that one can use to construct compact-object solutions** (single or multiple-horizon black holes, regular black holes, and solitons) with primary scalar hair in shift and $\mathbf{Z}_2$ symmetric beyond Horndeski theory.

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  The key ingredients of the method are:

  - A linearly time-dependent scalar field that carries the scalar hair.
  - The proportionality of the model functions $G_4(X) \propto G_2(X)$ and their power expansion

$$G_2(X) = \sum_{n=0}^{\infty} c_{\frac{n}{s}} X^{\frac{n}{s}} , \quad s \in \mathbb{Z}^+ .$$

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- We have identified the scalar charge/hair accompanying the solutions through the Noether current that emanates from the internal shift symmetry of the theory.

$$\left\{ \mathbf{J} = \mathcal{J}_\mu dx^\mu \,, \quad \mathcal{J}^\mu = \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta(\partial_\mu \phi)} = (\mathcal{J}^t, 0, 0, 0) \right\} \Rightarrow \boxed{Q_s = \int \star \mathbf{J} \propto q^k}$$

## Conclusions:

- We have demonstrated a **generic method that one can use to construct compact-object solutions** (single or multiple-horizon black holes, regular black holes, and solitons) with primary scalar hair in shift and $\mathbf{Z}_2$ symmetric beyond Horndeski theory.

  The key ingredients of the method are:
  - A linearly time-dependent scalar field that carries the scalar hair.
  - The proportionality of the model functions $G_4(X) \propto G_2(X)$ and their power expansion

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- The beyond Horndeski gravity constitutes a more natural framework to describe the Bardeen solution.

## Future Directions:

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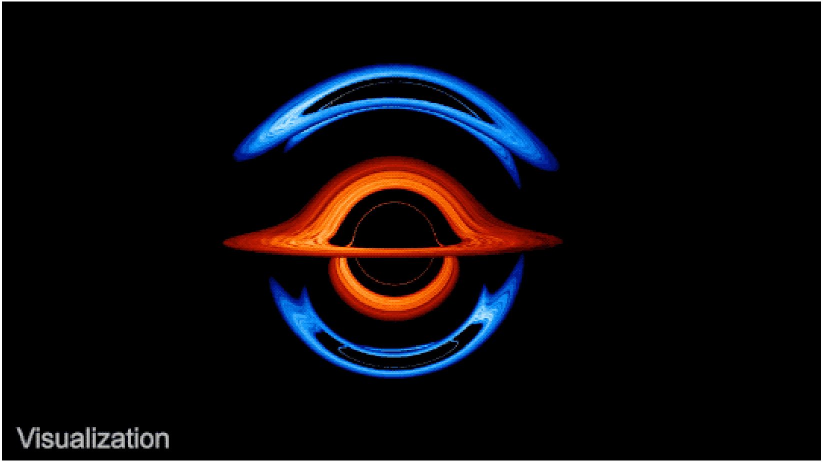
- Study the quasinormal modes of black holes with primary scalar hair.
- Is it possible to generalize the method to construct rotating black-hole solutions with primary scalar hair?

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