STOCHASTIC FORMALISM IN A PERTURBATIVE REGIME

Tomotaka KURODA

Institute of SCIENCE TOKYO

Institute for Basic Science

with Diego Cruces and … (ongoing)

Prague Spring 2025: CAS - IBS CTPU-CGA - ISCT Workshop in Cosmology, Gravitation and Particle Physics

@FZU, Prague, April 8th, 2025

OUTLINE

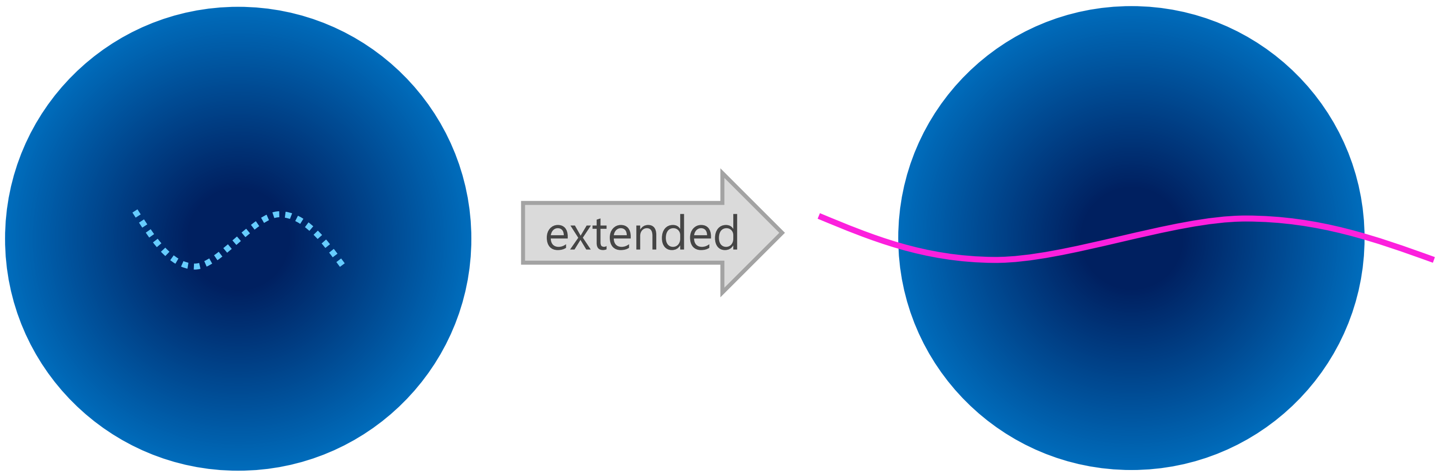
- Inflationary paradigm
- Stochastic formalism
- Correspondence to perturbation theory

INFLATIONARY PERTURBATION

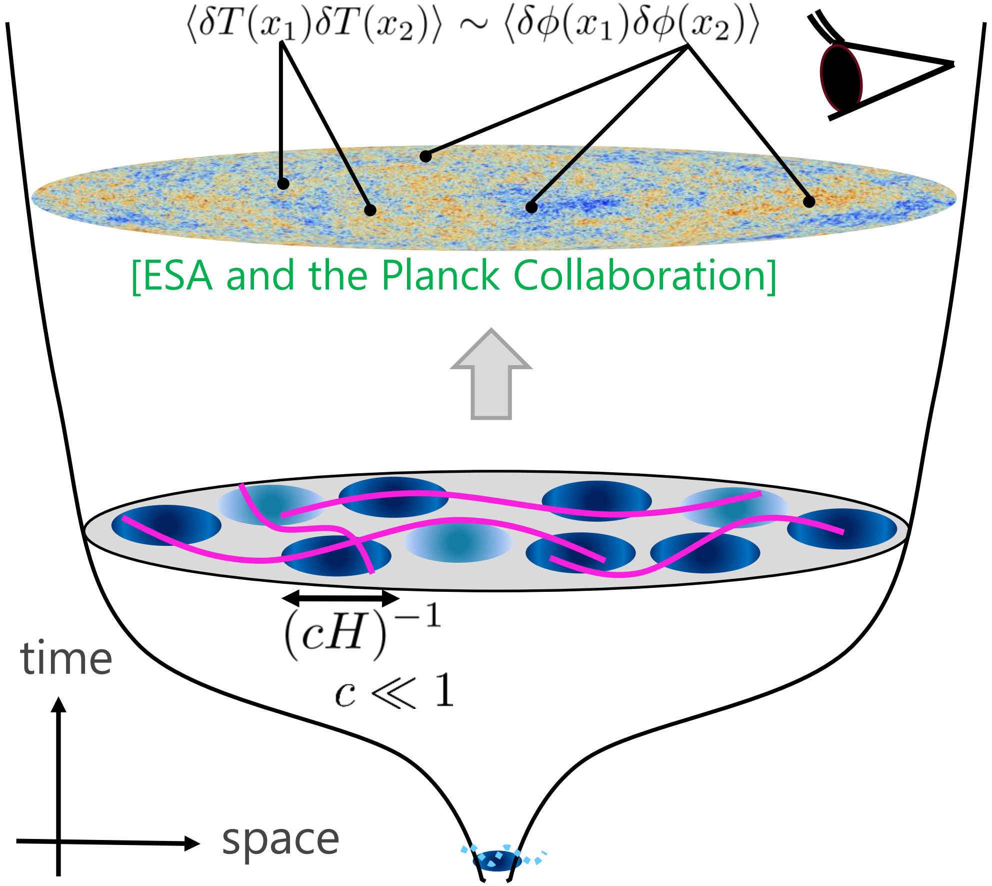
$H \equiv \dot{a}/a \Rightarrow H^{-1}$: characteristic length of the expansion

Statistical information (e.g. correlation function)

sub-horizon ***super***-horizon



Quantum to ***Classical***

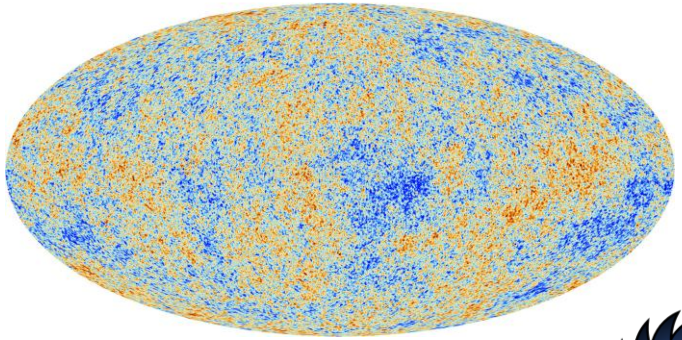


quantum fluctuations $\Rightarrow$ Classical IR modes $\Rightarrow$ inhomogeneity, anisotropy

FOOTPRINTS OF INFLATION

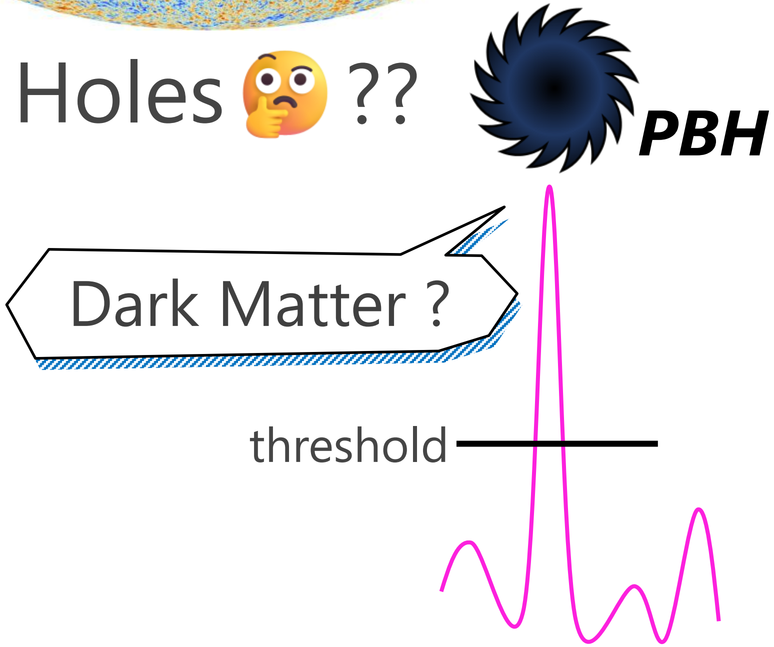
- Cosmic Microwave Background 😁 👍

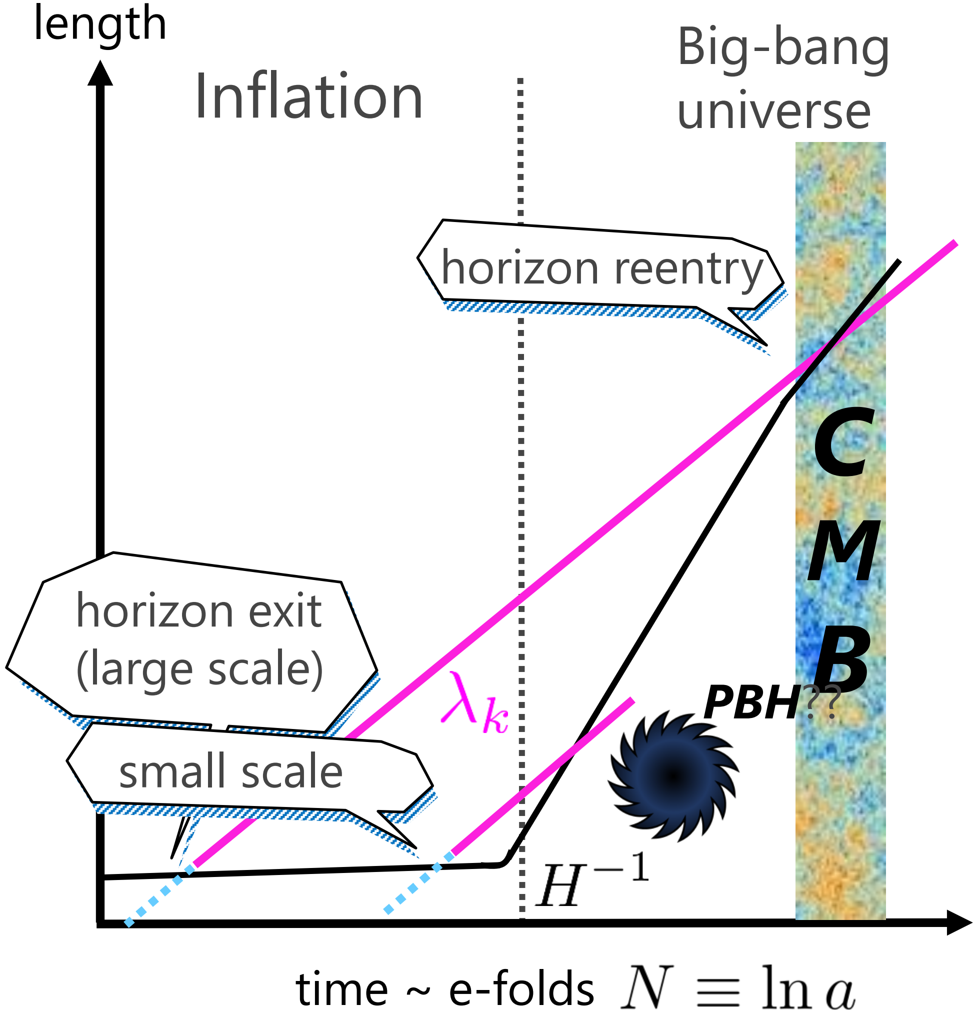
[ESA and the Planck Collaboration]



  - small amplitude
  - almost Gaussian

- Primordial Black Holes 🤔 ??
  - ***large*** amplitude
  - deviate from Gaussian





ISSUES TO BETACKLED

- Secular growth of correlation functions/ IR divergence

  Resummation of them
  by stochastic formalism with ***Fokker-Planck eq***, calculating PDFs directly

  [Starobinsky, et al astro-ph/9407016]

- Perturbations with a large amplitude

  necessary to take into account non-linearity, even more non-perturbativity

  Stochastic formalism with δN formalism; ***stochastic-δN program***

  [Fujita, et al 1308.4754]
  [Vennin, et al 1506.04732]

It remains unclear how it works concretely... ⇒ focusing on dynamical eqs
(especially when taking into account non-Markovianity)

OUTLINE

- Inflationary paradigm
- Stochastic formalism
- Correspondence to perturbation theory

STOCHASTIC FORMALISM

[Starobinsky (1986)]

- sub-horizon (UV) modes; QFT on curved spacetime

  ⇒ ***initial conditions*** with the (Bunch-Davies) vacuum

- super-horizon (IR) modes; classical, no causal contact…

  simplified

  ⇒ ***non-perturbative*** description (e.g. separate universe, δN formalism)

⇩

EFT for classical IR modes (for inflatons) including ***quantum corrections*** from UV modes by integrating them out

SEPARATE UNIVERSE

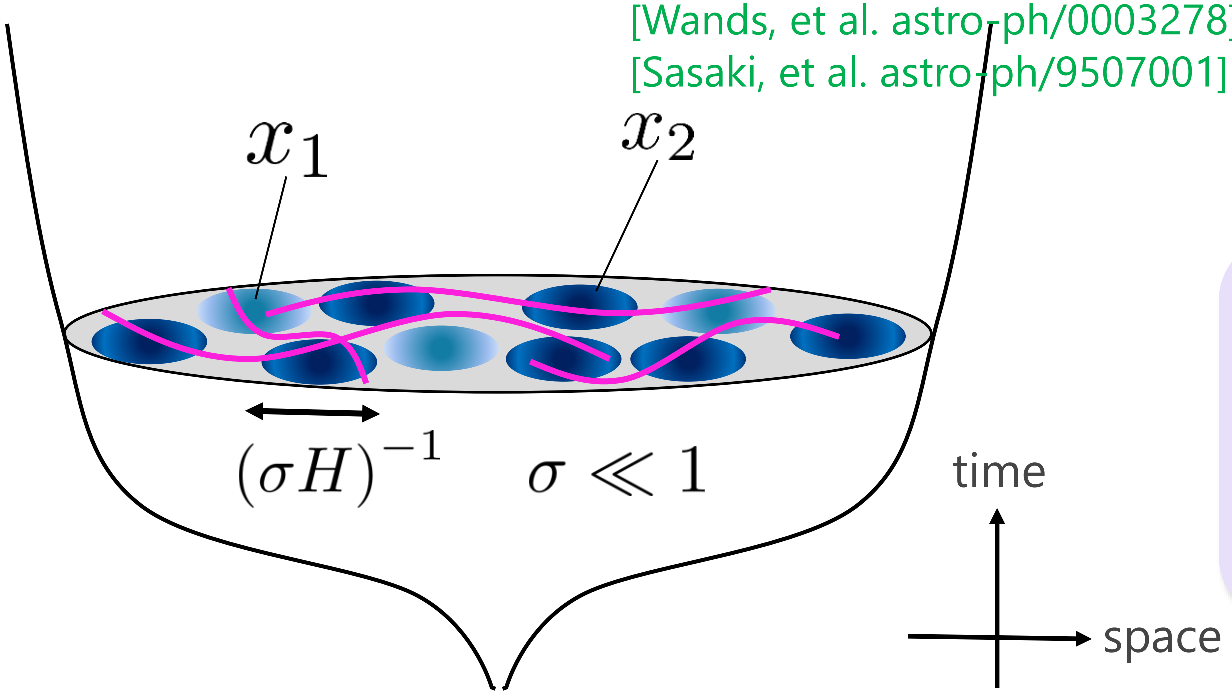
At super-horizon scale,

our universe

≀≀

an ensemble of patches,
independent FLRW universes

[Wands, et al. astro-ph/0003278]
[Sasaki, et al. astro-ph/9507001]



- Neglecting gradient terms

  e.g) Spatial derivative terms

- The evolution of inflatons

  ⇒ a KG equation in a local FLRW uni.

Stitching evolutions of local uni.

Perturbations on large scales

SEPARATION OF SCALES

$$\phi = \phi_{\mathrm{IR}} + \hat{\phi}_{\mathrm{UV}}$$ with ***time dep.*** coarse graining scale $k_\sigma(N) = \sigma a H(N)$

IR UV

$k_\sigma(N)$

- UV modes; ***quantum*** and ***free (linear)*** field with the BD vacuum

⇒ following Gaussian statics

$$\hat{\phi}_{\mathrm{UV}} = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \theta(k - k_\sigma(N)) \underbrace{\left(\phi_k \hat{a}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} + \phi_k^* \hat{a}_{\mathbf{k}}^\dagger e^{i\mathbf{k}\cdot\mathbf{x}} \right)}_{\equiv \hat{\phi}_{\mathbf{k}}}$$

- IR modes; ***classical*** field, which we would like to describe

$$\phi_{\mathrm{IR}} = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \left[1 - \theta(k - k_\sigma(N)) \right] \phi_{\mathbf{k}}$$

DYNAMICS FOR IR MODES

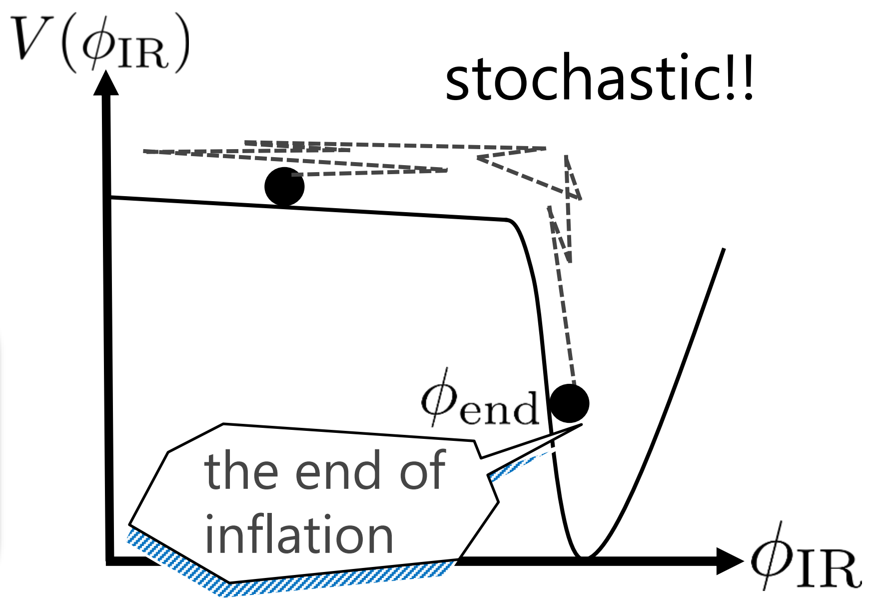
$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'(\phi)}{3H^2}$$

for one-patch

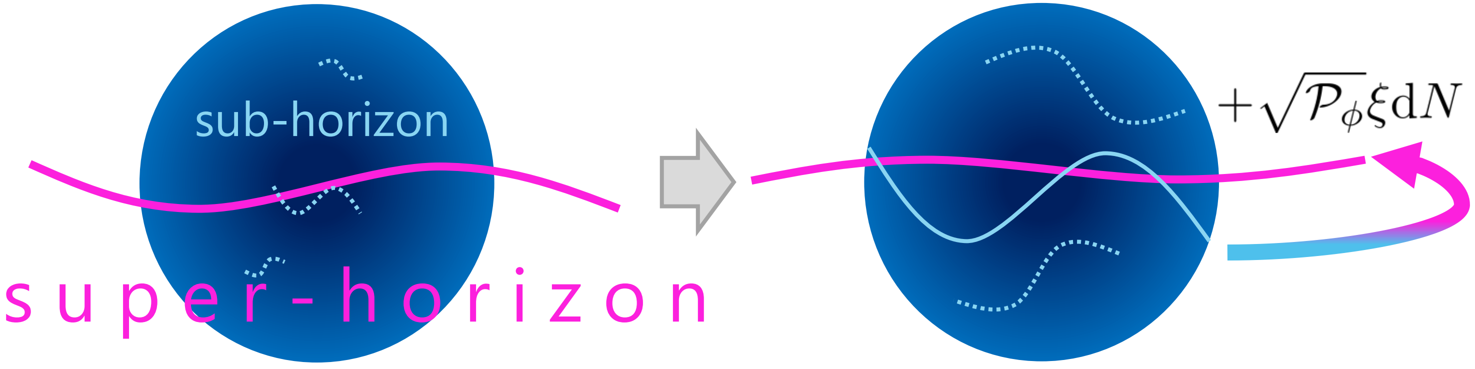
$$\phi = \phi_{\mathrm{IR}} + \hat{\phi}_{\mathrm{UV}}$$

$$\frac{\mathrm{d}\phi_{\mathrm{IR}}}{\mathrm{d}N} = -\frac{V'(\phi_{\mathrm{IR}})}{3H^2} + \sqrt{\mathcal{P}_\phi(N, k = k_\sigma(N))}\xi$$

non-perturbative evolution **: *Langevin eq.***



where $\sqrt{\mathcal{P}_\phi(N, k = k_\sigma(N))}\xi \equiv -\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\frac{\mathrm{d}k_\sigma(N)}{\mathrm{d}N}\delta(k - k_\sigma(N))\hat{\phi}_{\mathbf{k}}(N)$, $\quad \mathcal{P}_\phi = \frac{k^3}{2\pi^2}|\phi_{\mathrm{UV}}(k)|^2$



from sub-horizon to super-horizon at $k = k_\sigma(N)$

as ***Gaussian noise***

$$\langle\xi(N)\rangle = 0, \langle\xi(N)\xi(N')\rangle = \delta(N - N')$$

PROPERTIES OF NOISE

$\sqrt{\mathcal{P}_\phi(N, k = k_\sigma(N))}\xi$;evaluated at super-horizon crossing $k = k_\sigma(N)$

$\mathcal{P}_\phi = \frac{k^3}{2\pi^2}|\phi_{\rm UV}(k)|^2$ with a solution for EoM for free UV modes;

$$\frac{\partial^2\phi_{\rm UV}(k,N)}{\partial N^2} + 3\frac{\partial\phi_{\rm UV}(k,N)}{\partial N} + \left[\frac{k^2}{a^2H^2} + \frac{V''(\phi_{\rm IR})}{H^2} - \frac{1}{a^3HM_{\rm Pl}^2}\frac{\partial}{\partial N}\left(a^3H\left(\frac{\partial\phi_{\rm IR}}{\partial N}\right)^2\right)\right]\phi_{\rm UV}(k,N) = 0$$

over ***IR*** ***background***

(rather than deterministic or fixed background)

⇨ dependence on a whole history of an evolution of IR modes, and thus seemingly unsolvable (Generally speaking, $\frac{H}{2\pi}\xi$ is no longer correct)

⇨ ***non-Markovianity*** of the stochastic system

OUTLINE

- Inflationary paradigm
- Stochastic formalism
- Correspondence to perturbation theory

IN A PERTURBATIVE REGIME

In a perturbative regime, $\phi_{\rm IR}(t,\mathbf{x}) = \phi_{\rm IR}^{(0)}(t) + \phi_{\rm IR}^{(1)}(t,\mathbf{x}) + \phi_{\rm IR}^{(2)}(t,\mathbf{x}) + \cdots$

Langevin eq. : $\dfrac{{\rm d}\phi_{\rm IR}}{{\rm d}N} = -\dfrac{V'(\phi_{\rm IR})}{3H^2} + \underbrace{\sqrt{\mathcal{P}_\phi(N, k=k_\sigma(N))}}_{\equiv b(\phi_{\rm IR})}\xi$

$$\Leftrightarrow \quad \begin{aligned} &0^{\rm th};\ \frac{\partial \phi_{\rm IR}^{(0)}}{\partial N} = -\frac{V'(\phi_{\rm IR}^{(0)})}{3H^2} \\ &1^{\rm st};\ \frac{\partial \phi_{\rm IR}^{(1)}}{\partial N} = -\frac{V'(\phi_{\rm IR}^{(0)})}{3H^2}\phi_{\rm IR}^{(1)} + b(\phi_{\rm IR}^{(0)})\xi \\ &2^{\rm nd};\ \frac{\partial \phi_{\rm IR}^{(2)}}{\partial N} = -\frac{V'(\phi_{\rm IR}^{(0)})}{3H^2}\phi_{\rm IR}^{(2)} - \frac{V''(\phi_{\rm IR}^{(2)})}{3H^2}\frac{(\phi_{\rm IR}^{(1)})^2}{2!} + \frac{{\rm d}b(\phi_{\rm IR}^{(0)})}{{\rm d}\phi_{\rm IR}^{(0)}}\phi_{\rm IR}^{(1)}\xi \\ &\quad\vdots \end{aligned}$$

infinite sets of ***Wiener processes***!!

c.f) small noise expansion [Diego, et al. 2410.17987]

CORRESPONDENCE TO CPT

0th; $\frac{\partial \phi_{\mathrm{IR}}^{(0)}}{\partial N} = -\frac{V'(\phi_{\mathrm{IR}}^{(0)})}{3H^2}$ ⇨ ***EoM for 0th order perturbations*** on large scales in cosmological perturbation theory (background)

1st; $\frac{\partial \phi_{\mathrm{IR}}^{(1)}}{\partial N} = -\frac{V'(\phi_{\mathrm{IR}}^{(0)})}{3H^2}\phi_{\mathrm{IR}}^{(1)} + \underbrace{b(\phi_{\mathrm{IR}}^{(0)})}_{\equiv \frac{k^3}{2\pi^2}|\phi^{(1)}_{k\sim k_\sigma}|^2}\xi$ Fourier sp. ⇨ ***EoM for 1st order perturbations***

(In an opposite way of deriving Langevin eq from KG eq)

By definition

Sol of MS eq

With ***Gaussian distributed initial condition***

$\phi_{\mathrm{IR}}^{(1)}(N, k = k_\sigma(N))$

2nd; $\frac{\partial \phi_{\mathrm{IR}}^{(2)}}{\partial N} = -\frac{V'(\phi_{\mathrm{IR}}^{(0)})}{3H^2}\phi_{\mathrm{IR}}^{(2)} - \frac{V''(\phi_{\mathrm{IR}}^{(2)})}{3H^2}\frac{(\phi_{\mathrm{IR}}^{(1)})^2}{2!} + \underbrace{\frac{\mathrm{d}b(\phi_{\mathrm{IR}}^{(0)})}{\mathrm{d}\phi_{\mathrm{IR}}^{(0)}}}_{\frac{k^3}{2\pi^2}|\phi^{(2)}_{k\sim k_\sigma}|^2}\phi_{\mathrm{IR}}^{(1)}\xi$ Fourier sp. ⇨ ***EoM for 2nd order perturbations***

⋮

Can be proven (a new thing)

Sol of EoM2nd

CONCLUSION

- In a perturbative regime, (non-Markovian) Stochastic system can be recast into infinite sets of Wiener processes.

- The dynamical equation in stochastic inflation, i.e Langevin equation, is equivalent to EoMs of cosmological perturbation theory on large scale with Gaussian distributed initial condition. This means stochastic system can capture completely local type non-Gaussianity.

Future direction

- How non-perturbativity works
- intrinsic non-Gaussianity in UV modes
- beyond SR regime

Thank you for your attention!