Black holes and other exact solutions in six-derivative gravity

Breno L. Giacchini

Institute of Theoretical Physics, Charles University, Prague

Work in collaboration with Ivan Kolář: [2503.17318, 2406.00997]

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Higher derivatives & gravity

- General Relativity (GR)
  - Physical singularities for high curvatures.
  - General relativity is not perturbatively renormalizable.

$$d = 2p + 2$$

d: number of derivatives in the counterterms
p: order in the loop expansion

- Way out: GR is not valid at all scales.
- The “true” gravity theory is expect to resolve the singularities and coincide with GR in the low energy domain.

Higher derivatives & gravity

- Natural extension with higher-derivatives:
  - Higher-derivative terms appear already in QFT in curved spacetimes

$$R^2, \quad R^2_{\mu\nu}, \quad R^2_{\mu\nu\alpha\beta}, \quad \Box R$$

  - Fourth-order gravity is renormalizable [Stelle, 1977]

$$S_4 = \int d^4x \sqrt{-g}\left[\Lambda + \alpha R + \beta_1 R^2 + \beta_2 R^2_{\mu\nu} + \beta_3 R^2_{\mu\nu\alpha\beta} + \beta_4 \Box R\right]$$

  - Models with more than 4 derivatives are super-renormalizable [Asorey, López, Shapiro, 1997]

$$\begin{aligned} S_{2N+4} = \int d^4x \sqrt{-g}\Big[&\Lambda + \alpha R + \beta_1 R^2 + \beta_2 R^2_{\mu\nu} + \beta_3 R^2_{\mu\nu\alpha\beta} + \beta_4 \Box R \\ &+ \gamma_1 R \Box R + \gamma_2 R_{\mu\nu} \Box R^{\mu\nu} + \gamma_3 R_{\mu\nu\alpha\beta} \Box R^{\mu\nu\alpha\beta} + \gamma_4 R^3 + \ldots \\ &+ \rho_1 R \Box^N R + \rho_2 R_{\mu\nu} \Box^N R^{\mu\nu} + \rho_3 R_{\mu\nu\alpha\beta} \Box^N R^{\mu\nu\alpha\beta} + \rho_{\ldots} R^{N+2}_{\ldots}\Big] \end{aligned}$$

Higher derivatives & gravity

- Problem of ghosts: negative energy, violation of unitarity

$$\boxed{G_{\mathrm{spin}-2}(k) \sim \left(\frac{1}{k^2} - \frac{1}{k^2+m^2}\right)}$$

Renormalization of Higher Derivative Quantum Gravity

K.S. Stelle (Brandeis U.)
Oct, 1976

51 pages
Published in: *Phys.Rev.D* 16 (1977) 953-969
DOI: 10.1103/PhysRevD.16.953
Report number: PRINT-76-1059 (BRANDEIS)
View in: AMS MathSciNet, OSTI Information Bridge Server, ADS Abstract Service

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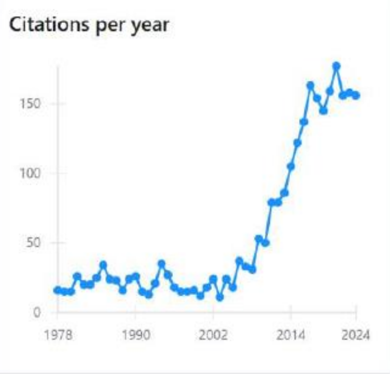
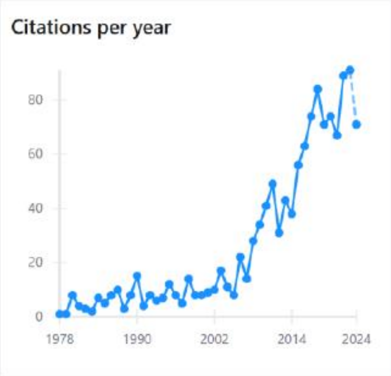


Classical Gravity with Higher Derivatives

K. S. Stelle (Brandeis U.)
Apr, 1977

34 pages
Published in: *Gen.Rel.Grav.* 9 (1978) 353-371
DOI: 10.1007/BF00760427
Report number: Print-77-0417 (BRANDEIS)
View in: AMS MathSciNet

cite claim reference search 1,250 citations



Higher derivatives & gravity

Perturbative quantum gravity & higher-derivative gravity revivals

- Effective approach to quantum gravity [Donoghue, 1994; Bjerrum-Bohr, Donoghue, Holstein, Planté, Vanhove, 2002, 2014, 2015,…]
- Renewed interest in non-local (ghost-free) quantum gravity [Modesto, 2012; Biswas, Gerwick, Koivisto & Mazumdar, 2012]
- New insights to tame (or understand) the ghosts:
  - Complex ghosts (Lee-Wick gravity) [Modesto & Shapiro, 2016; Asorey, Krein, Shapiro, 2024]
  - Quantization via *fakeons* [Anselmi & Piva, 2017]
  - Unstable ghosts [Donoghue & Menezes, 2019]
  - Stable and meta-stable scenarios with ghosts [Salvio & Strumia, 2016; Smilga, 2017; Reis, Chapiro & Shapiro, 2019; Donoghue & Menezes, 2021; Deffayet, Mukohyama & Vikman, 2022]
- (Renewed) interest on classical solutions

Exact solutions in higher-derivative gravity

- Mainly in quadratic gravity [Lü, Perkins, Pope & Stelle, 2015; Bonanno; Holdom; Podolský; Pravda; Pravdová; Ren; Švarc; Silveravalle;…]
- Exception I: Einsteinian cubic gravity and quasi-topological higher-dimensional models [Bueno & Cano, 2016; Hennigar & Mann, 2017; Hennigar, Kubizňák & Mann, 2017]
- Exception II: Holdom (2002), models with 4, 6, 8 and 10 derivatives.
  - Reported only finding regular solutions for general models with 6, 8 and 10 derivatives.

Exact solutions in sixth-derivative gravity

[BLG & Kolář, arXiv:2406.00997]

- General six-derivative gravity

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[\alpha R + \beta_1 R^2 + \beta_2 R_{\mu\nu}^2 + \gamma_1 R\Box R + \gamma_2 R_{\mu\nu}\Box R^{\mu\nu}$$
$$+ \gamma_3 R^3 + \gamma_4 R R_{\mu\nu}R^{\mu\nu} + \gamma_5 R_{\mu\nu}R^{\mu}{}_{\rho}R^{\rho\nu} + \gamma_6 R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma}$$
$$+ \gamma_7 R R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \gamma_8 R_{\mu\nu\rho\sigma}R^{\mu\nu\tau\upsilon}R^{\rho\sigma}{}_{\tau\upsilon}\Big],$$

- Static spherically symmetric solutions

$$\mathrm{d}s^2 = -B(r)\mathrm{d}t^2 + A(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

- Field equations

$$H_{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = 0 \qquad \nabla^{\nu}H_{\mu\nu} = 0$$

$$\boxed{H_{tt} = 0\,, \qquad H_{rr} = 0}$$

Frobenius series solutions around $r = 0$

$$A(r) = r^s \sum_{n=0}^{\infty} a_n r^n, \qquad \frac{B(r)}{b_0} = r^t \left(1 + \sum_{n=1}^{\infty} b_n r^n\right), \qquad s, t \in \mathbb{R}, \quad a_0, b_0 \neq 0$$

- Solution families are identified by the labels $(s, t)_0$, which are assumed to be independent of the couplings.
- System of indicial equations (lowest-order term of the field equations)

$$\frac{H_{tt}(r)}{B(r)} = r^{p(s)} \sum_{i=1}^{8} \gamma_i\, g_i(s, t, a_0) + \ldots = 0, \qquad H_{rr}(r) = r^{q(s)} \sum_{i=1}^{8} \gamma_i\, h_i(s, t, a_0) + \ldots = 0$$

$$g_i(s, t, a_0) = h_i(s, t, a_0) = 0, \quad \forall\, i = 1, \ldots, 8.$$

$$
\begin{aligned}
g_1(s,t) &= 4(s+2)[s(t+4)-t(t+2)-4][(6s-t)(10s^2-st-t^2)+2(118s^2-34st-t^2)+4(69s-14t+22)],\\
h_1(s,t) &= 4(s+2)[s(t+4)-t(t+2)-4][t(10s^2-st-t^2)+2(20s^2+6st-3t^2)+4(31s-6t+14)],\\
g_2(s,t) &= 2[st(60s^4-76s^3t+11s^2t^2+6st^3-t^4)+2(60s^5+146s^4t-208s^3t^2+35s^2t^3+8st^4-t^5+328s^4+152s^3t\\
&\quad -263s^2t^2+96st^3-3t^4)+8(124s^3-69s^2t-15st^2+21t^3-5s^2-110st+2t^2)-96(9s+2t+3)],\\
h_2(s,t) &= 2[st^2(10s^3-11s^2t+t^3)+2t(20s^4+6s^3t-21s^2t^2+2st^3+t^4)+2(60s^4+56s^3t-71s^2t^2-4st^3+21t^4)\\
&\quad +8(64s^3-9s^2t-33st^2-7t^3+47s^2-26st+6t^2)+96(2t-5s-3)],\\
g_3(s,t) &= 4[s(t+4)-t(t+2)-4]^2[30s^2+st-t^2+2(47s-t+37)],\\
h_3(s,t) &= 4[s(t+4)-t(t+2)-4]^2[t(5s+t)+2(10s+4t+23)],\\
g_4(s,t) &= 2[t^2(s-t)^2(30s^2+st-t^2)+2t(80s^4-55s^3t-71s^2t^2+47st^3-t^4)+2(140s^4+146s^3t-193s^2t^2+22st^3\\
&\quad +25t^4)+8(59s^3-20s^2t+23st^2+12t^3-39s^2+28st+56t^2)-32(8s-21t-19)],\\
h_4(s,t) &= 2[t^3(s-t)^2(5s+t)+8t^2(5s^3-6s^2t+t^3)+2t(70s^3-35s^2t-34st^2+27t^3)+8(30s^3+5s^2t-23st^2+14t^3\\
&\quad +19s^2-28st+48t^2)-32(21s-16t-17)],\\
g_5(s,t) &= t^2(s-t)^2(30s^2+st-t^2)+12st(10s^3-3s^2t-14st^2+7t^3)+6(40s^4+84s^3t-23s^2t^2-8st^3+7t^4)\\
&\quad +8(71s^3+69s^2t+24st^2+8t^3)+24(6s^2+16st+11t^2)-32(3s-9t-10),\\
h_5(s,t) &= t^3(s-t)^2(5s+t)+6t^2(5s^3-5s^2t-st^2+t^3)+6t(20s^3+3s^2t-12st^2+9t^3)+8(25s^3+27s^2t-3st^2\\
&\quad -5t^3)+24(10s^2+8st+9t^2)-32(9s-15t-14),\\
g_6(s,t) &= t^2(s-t)^2(30s^2+st-t^2)+2t(40s^4+19s^3t-97s^2t^2+37st^3+t^4)+4(20s^4+24s^3t-22s^2t^2-43st^3\\
&\quad +11t^4)+8(10s^3-5s^2t+23st^2-10t^3-25s^2+54st+39t^2)-32(5s-18t-7),\\
h_6(s,t) &= t^3(s-t)^2(5s+t)+4t^2(5s^3-3s^2t-3st^2+t^3)+4st(10s^2-9st-3t^2)+8(20s^3-7s^2t-14st^2+19t^3\\
&\quad +11s^2-14st+21t^2)-32(15s-4t-11),\\
g_7(s,t) &= 4[t^2(s-t)^2(30s^2+st-t^2)+2t(40s^4+19s^3t-97s^2t^2+37st^3+t^4)+2(40s^4+128s^3t+43s^2t^2-92st^3\\
&\quad +21t^4)+16(15s^3-3s^2t+27st^2-4t^3)+8(46s^2-36st+45t^2)+32(15s+13)],\\
h_7(s,t) &= 4[t^3(s-t)^2(5s+t)+4t^2(5s^3-3s^2t-3st^2+t^3)+2t(20s^3+17s^2t-20st^2+11t^3)+8(20s^3-2s^2t\\
&\quad +5st^2-t^3+26s^2-8st+31t^2)-32(6s-3t-11)],\\
g_8(s,t) &= 4[t^2(s-t)^2(30s^2+st-t^2)+6t^2(31s^3-41s^2t+9st^2+t^3+70s^2-54st+4t^2)-8(s^3-66st^2+19t^3)\\
&\quad +32(9t^2+1)],\\
h_8(s,t) &= 4[t^3(s-t)^2(5s+t)+24st^3(s-t)+12t^3(3s-t)+8(10s^3-3s^2t+7t^3)+144s^2+160].
\end{aligned}
$$

Frobenius series solutions around $r = 0$

$$A(r) = r^s \sum_{n=0}^{\infty} a_n r^n, \qquad \frac{B(r)}{b_0} = r^t \left(1 + \sum_{n=1}^{\infty} b_n r^n\right), \qquad s,t \in \mathbb{R}\,, \quad a_0, b_0 \neq 0$$

- Solution families are identified by the labels $(s,t)_0$, which are assumed to be independent of the couplings.
- System of indicial equations (lowest-order term of the field equations)

$$\frac{H_{tt}(r)}{B(r)} = r^{p(s)} \sum_{i=1}^{8} \gamma_i\, g_i(s,t,a_0) + \ldots = 0, \qquad H_{rr}(r) = r^{q(s)} \sum_{i=1}^{8} \gamma_i\, h_i(s,t,a_0) + \ldots = 0$$

$$g_i(s,t,a_0) = h_i(s,t,a_0) = 0, \quad \forall\, i = 1,\ldots,8.$$

- Only one family of solutions: $s = t = 0, \qquad a_0 = 1.$

- Only solution family: $(0,0)_0$, characterized by 6 parameters (5 physical)

$$A(r) = 1 + a_2 r^2 + a_3 r^3 + a_4 r^4 + \frac{(a_3 + b_3)\bar{a}_5}{\gamma_2(3\gamma_1 + \gamma_2)} r^5 + O(r^6),$$

$$\frac{B(r)}{b_0} = 1 + b_2 r^2 + b_3 r^3 + b_4 r^4 + \frac{(a_3 + b_3)\bar{b}_5}{\gamma_2(3\gamma_1 + \gamma_2)} r^5 + O(r^6)$$

with $(8\gamma_1 + 3\gamma_2)a_3 = 3(4\gamma_1 + \gamma_2)b_3$

- All solutions in this family are regular at $r = 0$:

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \underset{r\to 0}{=} 12\left(a_2^2 + b_2^2\right) + O(r)$$

- The solution explicitly requires $\gamma_2(3\gamma_1+\gamma_2) \neq 0$, *i.e.*, the model propagates two massive modes of spin-0 and of spin-2.
- No non-trivial solution such that $R = 0$.

❖ Solutions of incomplete (but renormalizable) six-derivative models

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\Big[\boxed{\alpha R + \beta_1 R^2 + \beta_2 R_{\mu\nu}^2 + \gamma_1 R\Box R + \gamma_2 R_{\mu\nu}\Box R^{\mu\nu}}$$
$$+ \gamma_3 R^3 + \gamma_4 R R_{\mu\nu}R^{\mu\nu} + \gamma_5 R_{\mu\nu}R^{\mu}{}_{\rho}R^{\rho\nu} + \gamma_6 R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma}$$
$$+ \gamma_7 R R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \gamma_8 R_{\mu\nu\rho\sigma}R^{\mu\nu\tau\upsilon}R^{\rho\sigma}{}_{\tau\upsilon}\Big],$$

Sixth-derivative terms in the action	(s,t) solution family	Number of free parameters
$R\Box R,\ R_{\mu\nu}\Box R^{\mu\nu},\ R_{\mu\nu\alpha\beta}\Box R^{\mu\nu\alpha\beta}$	$(0,0)$	$6 \to 5$
$R\Box R,\ R_{\mu\nu}\Box R^{\mu\nu}$	$(0,0)$ $(1,-1)$	$6 \to 5$ $3 \to 2$
$R\Box R,\ R_{\mu\nu\alpha\beta}\Box R^{\mu\nu\alpha\beta}$	$(0,0)$	$6 \to 5$
$R\Box R,\ C_{\mu\nu\alpha\beta}\Box C^{\mu\nu\alpha\beta}$	$(0,0)$ $(2,2)$	$6 \to 5$ $8 \to 7$

Solutions around other points

[BLG & Kolář, arXiv:2503.17318]

$$\boxed{\mathrm{d}s^2 = -H(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{H(r)} + F^2(r)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)}$$

$$\bar{r} = F(r)$$

$$A(\bar{r}) = \frac{1}{H(r)F'^2(r)},$$

$$B(\bar{r}) = H(r)$$

$$\mathrm{d}s^2 = -B(\bar{r})\mathrm{d}t^2 + A(\bar{r})\mathrm{d}\bar{r}^2 + \bar{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

Solutions around other points

[BLG & Kolář, arXiv:2503.17318]

$$\boxed{ds^2 = -H(r)dt^2 + \frac{dr^2}{H(r)} + F^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right)}$$

- Residual gauge freedom: $t \longrightarrow t' = \lambda t, \qquad r \longrightarrow r' = \lambda^{-1} r + \nu$
- Field equations do not depend explicitly on r.
- Solutions around $r = r_0$:

$$F(r) = \Delta^\sigma \sum_{n=0}^{\infty} f_n \Delta^n, \qquad H(r) = \Delta^\tau \sum_{n=0}^{\infty} h_n \Delta^n, \quad \text{with} \quad \Delta \equiv r - r_0 \quad \text{and} \quad f_0, h_0 \neq 0.$$

- Solutions around $r = \infty$: $F(r) = r^\sigma \sum_{n=0}^{\infty} f_n r^{-n}, \qquad H(r) = r^\tau \sum_{n=0}^{\infty} h_n r^{-n}$
- Solution families are identified by the labels $\{\sigma, \tau\}$ and $\{\sigma, \tau\}^\infty$
- Possible families of solutions: $\boxed{\{1,0\}, \quad \{0,0\}, \quad \{0,1\}, \quad \{0,2\},}$

$$\boxed{\{1,0\}^\infty, \quad \{1,2\}^\infty, \quad \{0,2\}^\infty.}$$

$$F(r) = \Delta^{\sigma} \sum_{n=0}^{\infty} f_n \Delta^n, \qquad H(r) = \Delta^{\tau} \sum_{n=0}^{\infty} h_n \Delta^n, \quad \text{with} \quad \Delta \equiv r - r_0 \quad \text{and} \quad f_0, h_0 \neq 0.$$

Solution family {1,0}: corresponds to expansions around the origin in standard spherically symmetric coordinates.

- Solution of field equations at lowest orders fixes $f_0^2 h_0 = 1$, $f_1 = h_1 = 0$ and

$$4(8\gamma_1 + 3\gamma_2) f_3 = -(10\gamma_1 + 3\gamma_2) f_0^3\, h_3.$$

- Starting at order Δ^{-1} (N=–1) it is possible to obtain recursive relations between the coefficients.

Solution characterized by 7 (5) free parameters: f_0, f_2, f_3(or h_3), f_4, h_2, h_4 and r_0.

$$F(r) = \Delta^{\sigma} \sum_{n=0}^{\infty} f_n \Delta^n, \qquad H(r) = \Delta^{\tau} \sum_{n=0}^{\infty} h_n \Delta^n, \quad \text{with} \quad \Delta \equiv r - r_0 \quad \text{and} \quad f_0, h_0 \neq 0.$$

<u>Solution family {0,0}:</u> corresponds to expansions around a generic point in standard spherically symmetric coordinates.

- From the lowest order Δ^0 (N=0) it is possible to obtain recursive relations between the coefficients:

$$f_{N+6} = -\frac{(2\gamma_1 + \gamma_2)\Phi_{\theta\theta,N+5} + (4\gamma_1 + \gamma_2)\Phi_{tt,N+5}}{4\gamma_2(3\gamma_1 + \gamma_2)h_0}, \qquad h_{N+6} = \frac{(4\gamma_1 + \gamma_2)\Phi_{\theta\theta,N+5} + (8\gamma_1 + 3\gamma_2)\Phi_{tt,N+5}}{2\gamma_2(3\gamma_1 + \gamma_2)f_0}.$$

- But, at lowest order we also have to solve the equation for the component *rr*, which yields a constraint between the parameters $f_{0,\ldots,5}, h_{0,\ldots,5}$:

$$120\left[2f_1h_0(8\gamma_1 + 3\gamma_2) + f_0h_1(4\gamma_1 + \gamma_2)\right]\frac{h_0^2}{f_0}f_5 + 60\left[2f_1h_0(4\gamma_1 + \gamma_2) + f_0h_1(2\gamma_1 + \gamma_2)\right]h_0\,h_5 - \frac{c}{f_0^5} = 0.$$

Solution characterized by 12 (10) free parameters.

$$F(r) = \Delta^{\sigma} \sum_{n=0}^{\infty} f_n \Delta^n, \qquad H(r) = \Delta^{\tau} \sum_{n=0}^{\infty} h_n \Delta^n, \quad \text{with} \quad \Delta \equiv r - r_0 \quad \text{and} \quad f_0, h_0 \neq 0.$$

Sub-cases of the solution family {0,0}:

- If some parameters $f_{0,\ldots,5}$ are set to 0, the solution can correspond to wormholes and non-Frobenius solutions in standard spherically symmetric coordinates:

$$ds^2 = B(\bar{r})\mathrm{d}t^2 + \frac{\mathrm{d}\bar{r}^2}{C(\bar{r})} + \bar{r}^2\mathrm{d}\Omega^2$$

For example:

$$\{0,0\}_{f_1=0, f_2\neq 0} \qquad \begin{aligned} C(\bar{r}) &= c_0\bar{\Delta} + c_1\bar{\Delta}^{3/2} + c_2\bar{\Delta}^2 + c_3\bar{\Delta}^{5/2} + c_4\bar{\Delta}^3 + O(\bar{\Delta}^{7/2}), \\ \frac{B(\bar{r})}{b_0} &= 1 + b_1\bar{\Delta}^{1/2} + b_2\bar{\Delta} + b_3\bar{\Delta}^{3/2} + b_4\bar{\Delta}^2 + b_5\bar{\Delta}^{5/2} + b_6\bar{\Delta}^3 + O(\bar{\Delta}^{7/2}). \end{aligned}$$

$$\{0,0\}_{f_{1,3,5}=h_{1,3,5}=0, f_2\neq 0} \qquad \begin{aligned} C(\bar{r}) &= c_0\bar{\Delta} + c_2\bar{\Delta}^2 + c_4\bar{\Delta}^3 + O(\bar{\Delta}^4), \\ \frac{B(\bar{r})}{b_0} &= 1 + b_2\bar{\Delta} + b_4\bar{\Delta}^2 + b_6\bar{\Delta}^3 + O(\bar{\Delta}^4). \end{aligned}$$

$$F(r) = \Delta^{\sigma} \sum_{n=0}^{\infty} f_n \Delta^n, \qquad H(r) = \Delta^{\tau} \sum_{n=0}^{\infty} h_n \Delta^n, \quad \text{with} \quad \Delta \equiv r - r_0 \quad \text{and} \quad f_0, h_0 \neq 0.$$

Solution family {0,1}: corresponds to expansions around a horizon in standard spherically symmetric coordinates.

- From the lowest order Δ^0 (N=0) it is possible to obtain recursive relations between the coefficients:

$$f_{N+3} = -\frac{(2\gamma_1 + \gamma_2)\Phi_{\theta\theta,N+2} - (4\gamma_1 + \gamma_2)\Phi_{tt,N+2}}{4\gamma_2(3\gamma_1 + \gamma_2)(N + 3)h_0}, \qquad h_{N+3} = \frac{(4\gamma_1 + \gamma_2)\Phi_{\theta\theta,N+2} - (8\gamma_1 + 3\gamma_2)\Phi_{tt,N+2}}{2\gamma_2(3\gamma_1 + \gamma_2)(N + 4)f_0}.$$

Solution characterized by 7 (5) free parameters.

$$F(r) = \Delta^\sigma \sum_{n=0}^{\infty} f_n \Delta^n, \qquad H(r) = \Delta^\tau \sum_{n=0}^{\infty} h_n \Delta^n, \quad \text{with} \quad \Delta \equiv r - r_0 \quad \text{and} \quad f_0, h_0 \neq 0.$$

Solution family {0,2}: corresponds to expansions around a double (degenerate) horizon in standard spherically symmetric coordinates.

- Several sub-families of solutions, with only two free parameters (h_1 and r_0). No physical free parameter.

$$\begin{aligned}
h_0 = -\frac{1}{f_0^2} \qquad &\Longrightarrow \qquad f_0^2 = \sqrt{c_1}, \\
h_0 = \frac{1}{f_0^2} \qquad &\Longrightarrow \qquad f_0^2 = \sqrt{c_2}, \\
h_0 = \frac{1}{f_0^2} + \frac{2\left(c_3 - \sqrt{c_4}\right)}{c_3^2 - c_4} \qquad &\Longrightarrow \qquad f_0^2 = -\frac{1}{2\sqrt{c_4}}\left[c_2 \pm \sqrt{c_2\left(c_2 - 2c_4 - 2c_3\sqrt{c_4}\right)}\right], \\
h_0 = \frac{1}{f_0^2} + \frac{2\left(c_3 + \sqrt{c_4}\right)}{c_3^2 - c_4} \qquad &\Longrightarrow \qquad f_0^2 = \frac{1}{2\sqrt{c_4}}\left[c_2 \pm \sqrt{c_2\left(c_2 - 2c_4 + 2c_3\sqrt{c_4}\right)}\right],
\end{aligned}$$

$$F(r) = \Delta^\sigma \sum_{n=0}^{\infty} f_n \Delta^n, \qquad H(r) = \Delta^\tau \sum_{n=0}^{\infty} h_n \Delta^n, \quad \text{with} \quad \Delta \equiv r - r_0 \quad \text{and} \quad f_0, h_0 \neq 0.$$

<u>Solution family {0,2}:</u> corresponds to expansions around a double (degenerate) horizon in standard spherically symmetric coordinates.

- Several sub-families of solutions, with only two free parameters (h_1 and r_0). No physical free parameter.

- Near extreme horizon geometries:

$$\mathrm{d}s^2 = -h_0(r-r_0)^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{h_0(r-r_0)^2} + f_0^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

  - Nariai
  - Bertotti-Robinson
  - Generalizations

Solution family $(s,t)_0$ or $(-s,t)_{\bar{r}_0}$	$\{\sigma,\tau\}$	Parameters	Number of free parameters
$(0,0)_0$	$\{1,0\}$	$f_0,\ f_2,\ (f_3,\ h_3),\ f_4,\ h_2,\ h_4,\ r_0$	$7 \to 5$
$(0,0)_{\bar{r}_0}$	$\{0,0\}$	$(f_0,\ f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ h_0,\ h_1,\ h_2,\ h_3,\ h_4,\ h_5),\ r_0$	$12 \to 10$
$(1,0)_{\bar{r}_0,1/2}$	$\{0,0\}_{f_1=0,\ f_2\neq 0}$	$(f_0,\ f_2,\ f_3,\ f_4,\ f_5,\ h_0,\ h_1,\ h_2,\ h_3,\ h_4,\ h_5),\ r_0$	$11 \to 9$
$(1,0)_{\bar{r}_0}$	$\{0,0\}_{f_{1,3,5}=h_{1,3,5}=0,\ f_6\neq 0}$	$(f_0,\ f_2,\ f_4,\ h_0,\ h_2,\ h_4),\ r_0$	$6 \to 4$
$\left(\frac{4}{3},0\right)_{\bar{r}_0,1/3}$	$\{0,0\}_{f_{1,2}=0,\ f_3\neq 0}$	$(f_0,\ f_3,\ f_4,\ f_5,\ h_0,\ h_1,\ h_2,\ h_3,\ h_4,\ h_5),\ r_0$	$10 \to 8$
$\left(\frac{3}{2},0\right)_{\bar{r}_0,1/4}$	$\{0,0\}_{f_{1,2,3}=0,\ f_4\neq 0}$	$(f_0,\ f_4,\ f_5,\ h_0,\ h_1,\ h_2,\ h_3,\ h_4,\ h_5),\ r_0$	$9 \to 7$
$\left(\frac{3}{2},0\right)_{\bar{r}_0,1/2}$	$\{0,0\}_{f_{1,2,3,5}=h_{1,3,5}=0,\ f_6\neq 0}$	$(f_0,\ f_4,\ h_0,\ h_2,\ h_4),\ r_0$	$5 \to 3$
$\left(\frac{8}{5},0\right)_{\bar{r}_0,1/5}$	$\{0,0\}_{f_{1,2,3,4}=0,\ f_5\neq 0}$	$(f_0,\ f_5,\ h_0,\ h_1,\ h_2,\ h_3,\ h_4,\ h_5),\ r_0$	$8 \to 6$
$\left(\frac{5}{3},0\right)_{\bar{r}_0,1/6}$	$\{0,0\}_{f_{1,2,3,4,5}=0,\ f_6\neq 0}$	$(f_0,\ h_0,\ h_1,\ h_2,\ h_3,\ h_4,\ h_5),\ r_0$	$7 \to 5$
$\left(\frac{5}{3},0\right)_{\bar{r}_0,1/3}$	$\{0,0\}_{f_{1,2,3,4,5}=h_{1,3,5}=0,\ f_6\neq 0}$	$(f_0,\ h_0,\ h_2,\ h_4),\ r_0$	$4 \to 2$
$(1,1)_{\bar{r}_0}$	$\{0,1\}$	$f_0,\ f_1,\ f_2,\ h_0,\ h_1,\ h_2,\ r_0$	$7 \to 5$
$\left(\frac{3}{2},\frac{1}{2}\right)_{\bar{r}_0,1/2}$	$\{0,1\}_{f_1=0,\ f_2\neq 0}$	$f_0,\ f_2,\ h_0,\ h_1,\ h_2,\ r_0$	$6 \to 4$
$\left(\frac{5}{3},\frac{1}{3}\right)_{\bar{r}_0,1/3}$	$\{0,1\}_{f_{1,2}=0,\ f_3\neq 0}$	$f_0,\ h_0,\ h_1,\ h_2,\ r_0$	$5 \to 3$
$(2,2)_{\bar{r}_0}$	$\{0,2\}$	$(f_1,\ h_1),\ r_0$	$2 \to 0$
—	$\{0,2\}_{f_1=h_1=0}$	r_0	$1 \to 0$

$$F(r) = r^{\sigma} \sum_{n=0}^{\infty} f_n r^{-n}, \qquad H(r) = r^{\tau} \sum_{n=0}^{\infty} h_n r^{-n}$$

$$ds^2 = B(\bar{r})\mathrm{d}t^2 + \frac{\mathrm{d}\bar{r}^2}{C(\bar{r})} + \bar{r}^2 \mathrm{d}\Omega^2$$

Solution family $(-s,t)^{\infty}$	Solution family $\{\sigma,\tau\}^{\infty}$	Parameters	Number of free parameters	Interpretation
$(0,0)^{\infty}$	$\{1,0\}^{\infty}$	f_0, f_1, h_1	$3 \to 1$	corrections to Schwarzschild
$(0,0)^{\infty}$	$\{1,0\}^{\infty}_{h_1=0}$	f_0, f_1	$2 \to 0$	Minkowski
$(2,2)^{\infty}$	$\{1,2\}^{\infty}$	f_0, h_1, h_3	$3 \to 1$	corrections to Schwarzschild–(A)dS
$(2,2)^{\infty}$	$\{1,2\}^{\infty}_{h_3=0}$	f_0, h_1	$2 \to 0$	de Sitter or anti-de Sitter
—	$\{0,2\}^{\infty}$	h_1, h_2	$2 \to 0$	extreme near-horizon geometry

$$C(\bar{r}) = 1 + \frac{b_1}{\bar{r}} + \frac{18b_1^2(4\gamma_7 + 3\gamma_8)}{\alpha\bar{r}^6} + \frac{b_1^3(66\gamma_7 + 49\gamma_8)}{\alpha\bar{r}^7} + \frac{720b_1^2[8\gamma_7(3\beta_1 + \beta_2) + 3\gamma_8(4\beta_1 + \beta_2)]}{\alpha^2\bar{r}^8} + O(\bar{r}^{-9}),$$

$$\frac{B(\bar{r})}{b_0} = 1 + \frac{b_1}{\bar{r}} - \frac{12b_1^2\gamma_7}{\alpha\bar{r}^6} - \frac{b_1^3(18\gamma_7 + 5\gamma_8)}{\alpha\bar{r}^7} - \frac{180b_1^2[4\gamma_7(3\beta_1 + \beta_2) + 3\gamma_8(2\beta_1 + \beta_2)]}{\alpha^2\bar{r}^8} + O(\bar{r}^{-9}).$$

$$C(\bar{r}) = c_0\bar{r}^2 + 1 + \frac{c_3}{\bar{r}} + \frac{c_6}{\bar{r}^4} + O(\bar{r}^{-6}), \qquad \frac{B(\bar{r})}{b_0} = \bar{r}^2 + \frac{1}{c_0} + \frac{c_3}{c_0\bar{r}} + \frac{b_6}{\bar{r}^4} + O(\bar{r}^{-6})$$

$$\mathrm{d}s^2 = -(1 + h_0 r^2)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 + h_0 r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \qquad h_0 = \pm\sqrt{\frac{\alpha}{144\gamma_3 + 36\gamma_4 + 9\gamma_5 + 9\gamma_6 + 24\gamma_7 + 4\gamma_8}}$$

Conclusions

- Higher-derivative gravity models have many applications in quantum gravity, but the study of their classical solutions are still at early stages.

Exact static spherically symmetric solutions of six-derivative gravity:

- Solutions with Frobenius expansion around $r = 0$ (with exponents independent of the couplings) are necessarily regular.

	4-derivative gravity [Podolský, Švarc, Pravda, Pravdova, 2020]	6-derivative gravity
Solutions around $r = 0$	$(0{,}0),\ (1,-1),\ (2{,}2)$	$(0{,}0)$
Solutions around $r \neq 0$	$(0{,}0)_{r_0},\ (1{,}1)_{r_0},\ (1{,}0)_{r_0}$	$(0{,}0)_{r_0},\ (1{,}1)_{r_0},\ (1{,}0)_{r_0},\ (2{,}2)_{r_0}$
	+3 non-Frobenius families	+9 non-Frobenius families

[BLG & Kolář, arXiv:2503.17318]

Conclusions

Exact static spherically symmetric solutions of six-derivative gravity:

- We identified solutions with black hole horizons, extreme horizons and their near-horizon geometries.
  - This might be an indication that the models can have regular black hole solutions.
- Solutions that are asymptotically (A)dS, with effective cosmological constant

$$\Lambda_{\text{eff}} = \pm\sqrt{\frac{\alpha}{144\gamma_3 + 36\gamma_4 + 9\gamma_5 + 9\gamma_6 + 24\gamma_7 + 4\gamma_8}}.$$

- Conditions $R = 0$ and/or $g_{tt}g_{rr} = -1$ are too restrictive.

[BLG & Kolář, arXiv:2503.17318]

Antonín Slavíček: View of Prague from Ládví

Thank you!