Ghosts without runaway instabilities

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C.D., S. Mukohyama, A. Vikman,
PRL 128 (2022) 4, 041301.

C.D., A. Held, S. Mukohyama, A. Vikman
JCAP 11 (2023) 031.

C.D. A. Held, A. Jalali, S. Mukohyama, A. Vikman, in preparation.

I. Introduction

« Ghosts » are usually considered as degrees of freedom with « wrong sign » kinetic energies.

e.g. in a mechanical model, if a standard degree of freedom x has a free Hamiltonian

$$H = p_x^2$$

A ghost y would have the Hamiltonian

$$H = -\, p_y^2$$

In a scalar field theory, if a standard degree of freedom ϕ has a free Lagrangian

$$\mathrm{L} = \partial_\mu \phi \, \partial^\mu \phi$$

A ghost ψ would have the Lagrangian

$$\mathrm{L} = - \partial_\mu \psi \, \partial^\mu \psi$$

In field theories (with gravity), ghosts being associated with negative energies, there existence is linked with violations of energy conditions

- Free ghosts are not problematic

- E.g. a stable positive energy x with Hamiltonian

$$H_x = p_x^2 + \omega_x^2 \, x^2$$

Can coexist stably with a ghost y with Hamiltonian

$$H_y = -p_y^2 \, - \omega_y^2 \, y^2$$

- Or similarly, e.g.

$$\begin{aligned} H = &+ p_x^2 - \omega_x^2 \, x^2 + \Omega_x^4 \, x^4 \\ &- p_y^2 + \omega_y^2 \, y^2 - \Omega_y^4 \, y^4 \end{aligned}$$

- Free ghosts are not problematic

- The trouble arises when they interact with (standard) non ghost degrees of freedom

- Ghosts should at least interact gravitationnally with standard degrees of freedom

Standard lore:

These interactions lead to instabilities which can already be seen @ classical level in the form of runaways solutions.

NB: Various type of instabilities (stabilities) can be considered

@ Classical level

- Local stability :
  the motion stays close to its initial values

- Global stability :
  No runaway solutions: the motion stays bounded for all initial data

@ Quantum level

- No unitarity, vaccuum decay etc..

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Our papers

@ Quantum level

- No unitarity, vacuum decay etc..

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@ Classical level

- Local stability :
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- Global stability :
  No runaway solutions: the motion stays bounded for all initial data

This talk

@ Quantum level

- No unitarity, vacuum decay etc..

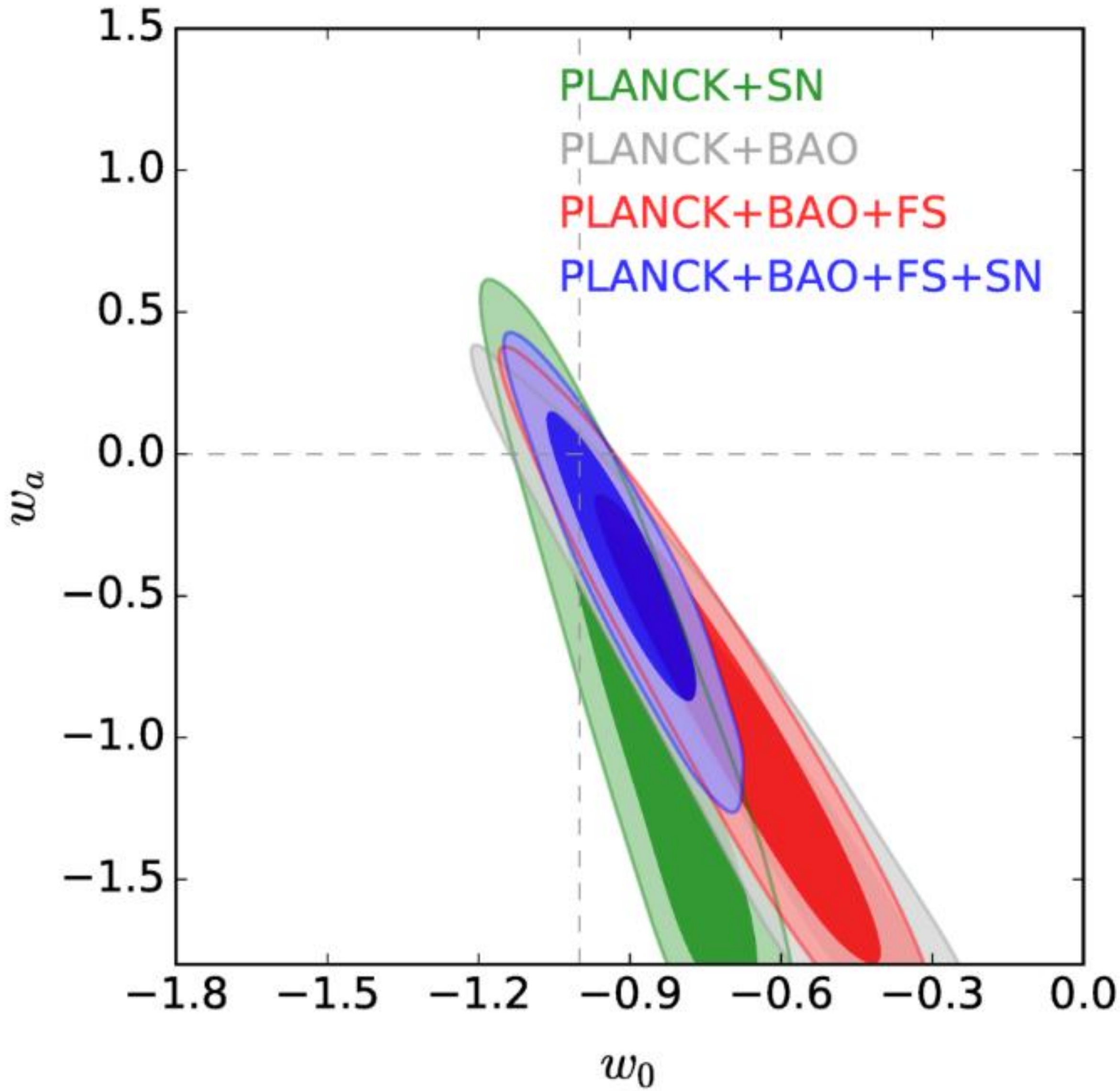
Yet, ghosts could have some interest to describe the real world

- Higher derivative theory, some with potentially interesting applications (e.g. Stelle 1976 for gravity, see also Lee, Wick, 1970) feature generically ghosts (Ostogradsky, 1850).

- Could be used to address the cosmological constant problem using ghostly copies of the standard (model) fields ? Linde 1984-1988, Kaplan-Sundrum 2006.

- To obtain bouncing cosmologies ? Brandenberger, Peter, 2017.

In cosmology: room for (dominant or weak) energy condition violations with dark energy (e.g. « Phantom DE » Caldwell, 2002)



$$w_{DE} = w_0 + \frac{z}{1+z}\, w_a$$

Constraints on w_0 and w_a from various observations including SNIa (SN) and CMB (PLANCK)

Figure taken from the PDG review on dark energy:

Or from DESI 2024-2025

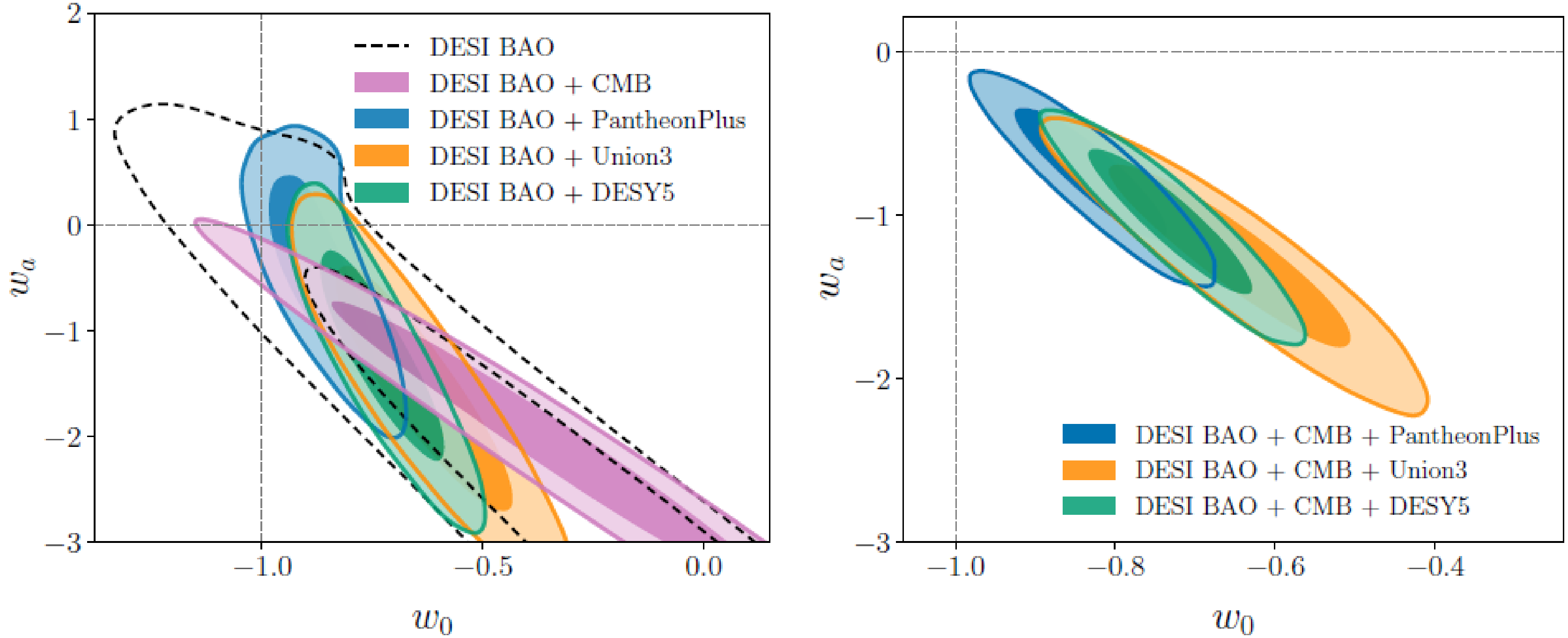


Figure 6. *Left panel*: 68% and 95% marginalized posterior constraints in the w_0–w_a plane for the flat w_0w_aCDM model, from DESI BAO alone (black dashed), DESI + CMB (pink), and DESI + SN Ia, for the PantheonPlus [24], Union3 [25] and DESY5 [26] SNIa datasets in blue, orange and green respectively. Each of these combinations favours $w_0 > -1$, $w_a < 0$, with several of them exhibiting mild discrepancies with ΛCDMat the $\gtrsim 2\sigma$ level. However, the full constraining power is not realised without combining all three probes. *Right panel*: the 68% and 95% marginalized posterior constraints from DESI BAO combined with CMB and each of the PantheonPlus, Union3 and DESY5 SN Ia datasets. The significance of the tension with ΛCDM ($w_0 = -1$, $w_a = 0$) estimated from the $\Delta\chi^2_{\rm MAP}$ values is 2.5σ, 3.5σ and 3.9σ for these three cases respectively.

- Some authors have argued that ghosts can be benign, even with runaways

  Smilga 2004, 2021; Damour, Smilga, 2021

- Some scarce numerical studies indicates that a ghost can stably (global stability) interact with a positive energy d.o.f. or have « islands » of stable initial conditions (also expected from Kolmogorov-Arnold-Moser (KAM) theorem).

  Pagani, Tecchiolli, Zerbini, 1987; Smilga 2005; Carrol, Hoffman, Trodden, 2003; Pavsic, 2016, 2013; Boulanger, Buisseret, Dierick, White, 2019.

II. Ghostifying (integrable) models :

building (integrable) theories of a ghost interacting with a positive energy degree of freedom

Start with a system with *N* positive energy degrees of freedom:

N canonical pairs $\xi_i = (x_i, p_i)$

And Hamiltonian

$$H(x_i, p_i) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(x_i, ...x_N)$$

Then consider the complex transformation

$$\mathfrak{C}^{\pm}_{x^n} : \left\{ \begin{array}{llll} x_j & \rightarrow & \pm i\, x_j & \text{for} \quad j = n \\ p_j & \rightarrow & \mp i\, p_j & \text{for} \quad j = n \\ x_j & \rightarrow & x_j & \quad \forall \quad j \neq n \\ p_j & \rightarrow & p_j & \quad \forall \quad j \neq n \end{array} \right\}$$

It preserves the Poisson brackets $\{\}_{\mathrm{PB}}$

$$\{\mathfrak{C}^{\pm}_{x^n}(x_i),\, \mathfrak{C}^{\pm}_{x^n}(p_j)\}_{\mathrm{PB}} = -\{\mathfrak{C}^{\pm}_{x^n}(p_i),\, \mathfrak{C}^{\pm}_{x^n}(x_j)\}_{\mathrm{PB}} = \delta_{ij} \ ,$$

$$\{\mathfrak{C}^{\pm}_{x^n}(x_i),\, \mathfrak{C}^{\pm}_{x^n}(x_j)\}_{\mathrm{PB}} = 0 \ ,$$

$$\{\mathfrak{C}^{\pm}_{x^n}(p_i),\, \mathfrak{C}^{\pm}_{x^n}(p_j)\}_{\mathrm{PB}} = 0 \ .$$

Then consider the complex transformation

$$\mathfrak{C}^{\pm}_{x^n} : \left\{ \begin{array}{lllll} x_j & \to & \pm i\, x_j & \text{for} & j = n \\ p_j & \to & \mp i\, p_j & \text{for} & j = n \\ x_j & \to & x_j & \forall & j \neq n \\ p_j & \to & p_j & \forall & j \neq n \end{array} \right\}$$

But flips the sign of the kinetic energy of the n^{th} d.o.f.

$$p_n \to \pm i p_n \quad \Rightarrow \quad p_n^2/m_n \to -p_n^2/m_n$$

Transforming the n^{th} positive energy (**P**) d.o.f.
into a ghosty (**G**) d.o.f.

E.g. consider a **two** degree of freedom system with positive kinetic energies and Hamiltonian

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V(x, y)$$

a « **PP** » system

Then, do the transformation $\mathfrak{C}_y^{\pm}$ $\begin{cases} y \to \pm i\, y \\ p_y \to \mp i\, p_y \end{cases}$

This transforms the system to a « **PG** » one

$$H = \frac{1}{2}p_x^2 - \frac{1}{2}p_y^2 + V(x, \pm iy)$$

The interaction $V(x,y)$

Under the transformation $\mathfrak{C}_y^{\pm}$ $\left\{ \begin{array}{l} y \rightarrow \pm i\, y \\ p_y \rightarrow \mp i\, p_y \end{array} \right.$

$V(x, \pm iy)$

Can become complex

But not always true: e.g. $x^2y^2 \rightarrow -x^2y^2$ etc....

The preservation of the Poisson brackets,
implies that an integrable PP system

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V(x, y)$$

I.e. such that there exists an extra integral of motion *I* (besides the Hamiltonian)

$$\frac{dI}{dt} = \{I, H\}_{\mathrm{PB}} = 0$$

Is transformed under $\mathfrak{C}_y^{\pm}$ $\left\{ \begin{array}{l} y \to \pm i\, y \\ p_y \to \mp i\, p_y \end{array} \right.$

To an integrable « PG » system (with the « same » *I*)

This can be used to build integrable stable PG systems out of integrable PP systems

Integrable (mostly PP) systems with 2 d.o.f. have been studied and classified since long ago …

… with the Pioneer work of Liouville 1855, Darboux 1901

With a Hamiltonian of the form

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V(x,y)$$

A (full ?) classification based on the **degree** of the integral motion I in the momenta exists up to … a not very large degree (2?)

(Darboux 1901; Whittaker 1964; Fris, Smorodinskii, Uhlir, Winternitz 1967; Holt 1982, Ankiewicz, Pask, 1983; Dorizzi, Grammaticos, Ramani, 1983; Thompson, 1984; Sen, 1985; Hietarinta, 1987; Nagakawa, Yoshida, 2001; Mitsopoulos, Tsamparlis, Paliathanasis, 2020).

At **linear** order (**in the momenta**) there is only one class of integrable PP systems given by

- Class 0 :

$$\begin{cases} V = f\left(\dfrac{a}{2}(x^2+y^2) + c\,x - b\,y\right) \,, \\ I = a\,(y\,p_x - x\,p_y) - b\,p_x - c\,p_y \end{cases}$$

At **quadratic** order (**in the momenta**), we have

- Class 1 :

$$\begin{cases} V = \dfrac{f(u) - g(v)}{u^2 - v^2}\,, \\ I = -\,(x p_y - y p_x)^2 - c\, p_x^2 \\ \qquad + 2\,\dfrac{u^2 g(v) - v^2 f(u)}{u^2 - v^2}\,, \end{cases}$$

With

$$\begin{cases} r^2 = x^2 + y^2 \\ u^2 = \dfrac{1}{2}\left(r^2 + c + \sqrt{(r^2+c)^2 - 4\,c\,x^2}\right) \\ v^2 = \dfrac{1}{2}\left(r^2 + c - \sqrt{(r^2+c)^2 - 4\,c\,x^2}\right) \end{cases}$$

f and *g* arbitrary functions

- Class 1 :

$$\begin{cases} V = \dfrac{f(u) - g(v)}{u^2 - v^2} \,, \\ I = -\,(xp_y - yp_x)^2 - c\,p_x^2 \\ \qquad + 2\,\dfrac{u^2 g(v) - v^2 f(u)}{u^2 - v^2} \,, \end{cases}$$

With

$$\begin{cases} r^2 = x^2 + y^2 \\ u^2 = \dfrac{1}{2}\left(r^2 + c + \sqrt{(r^2+c)^2 - 4\,c\,x^2}\right) \\ v^2 = \dfrac{1}{2}\left(r^2 + c - \sqrt{(r^2+c)^2 - 4\,c\,x^2}\right) \end{cases}$$

f and *g* arbitrary functions

This class 1 model (first discovered by Liouville, 1855) will play a central role in the rest of our work

There exists other classes with quadratic in momenta integral of motion I given by

- Class 2 :

$$\begin{cases} V = g(r) + \dfrac{f(x/y)}{r^2} , \\ I = (xp_y - yp_x)^2 + 2\, f(x/y) , \end{cases}$$

- Class 4 :

$$\begin{cases} V = \dfrac{f(r+y) + g(r-y)}{r} , \\ I = (yp_x - xp_y)p_x \end{cases}$$

$$+ \frac{(r+y)g(r-y) - (r-y)f(r+y)}{r} ,$$

With

$$\begin{cases} r^2 = x^2 + y^2 \\ f \text{ and } g \text{ arbitrary functions} \end{cases}$$

III. Stable motion of a ghost interacting with a positive energy degree of freedom

Consider now the « Liouville » class 1 integrable model

$$H_{\mathrm{LV}} = \frac{p_x^2}{2} + \sigma \frac{p_y^2}{2} + V_{\mathrm{LV}}(x, y)$$

With

$$V_{\mathrm{LV}} = \frac{f(u) - g(v)}{u^2 - v^2} \,,$$

$$u^2 = \frac{1}{2}\left(r^2 + c + \sqrt{(r^2+c)^2 - 4\,c\,x^2}\right) \,,$$

$$v^2 = \frac{1}{2}\left(r^2 + c - \sqrt{(r^2+c)^2 - 4\,c\,x^2}\right) \,,$$

$$r^2 = x^2 + \sigma\, y^2 \,.$$

f, g arbitrary functions, c an arbitrary constant

The original model is **PP** (*i.e.* $\boldsymbol{\sigma} = +\mathbf{1}$), but by using a proper complex canonical transformation, it can be transformed to a **PG** one (*i.e.* $\boldsymbol{\sigma} = -\mathbf{1}$).

The system has two constant of motion :

The Hamiltonian

$$\begin{cases} H_{\mathrm{LV}} = \dfrac{p_x^2}{2} + \sigma \dfrac{p_y^2}{2} + V_{\mathrm{LV}}(x, y) \\ V_{\mathrm{LV}} = \dfrac{f(u) - g(v)}{u^2 - v^2} \end{cases}$$

And

$$\begin{cases} I_{\mathrm{LV}} = -\sigma \left(p_y x - \sigma\, p_x y\right)^2 - c\, p_x^2 + \mathcal{V} \\ \mathcal{V} = 2\, \dfrac{u^2 g(v) - v^2 f(u)}{u^2 - v^2} \end{cases}$$

Now consider the PG case (with $\sigma = -1$)
and negative c (or positive $\widetilde{c} = -c$)

This makes the momentum-dependent part of I_{LV} positive

We prove (for the PG case with negative *c*) that the phase-space motion is bounded if

(i) $f(u)$ and $g(v)$ are bounded below, i.e.,

$$f(u) \geqslant f_0$$
$$g(v) \geqslant g_0$$

with constants $f_0, g_0 \in \mathbb{R}$; and

(ii) at large $|u|$ and $|v|$, these lower bounds sharpen to

$$f(u) \geqslant 4F_0\,|u|^{\zeta} > 0$$
$$g(v) \geqslant 4G_0\,|v|^{\eta} > 0$$

with positive constants $F_0, G_0 \in \mathbb{R}^+$ as well as $\zeta > 2$ and $\eta > 2$

Steps of the proof

Define a new first integral

$$J_{\mathrm{LV}} = I_{\mathrm{LV}} - \tilde{c}\, H_{\mathrm{LV}}$$
$$= \left(x p_y + y p_x\right)^2 + \frac{\tilde{c}}{2}\left(p_x^2 + p_y^2\right) + \mathcal{U}$$

Where

$$\mathcal{U}\left(u, v\right) = \frac{\left(2\,u^2 + \tilde{c}\right) g\left(v\right) + \left(2\,\tilde{v}^2 - \tilde{c}\right) f\left(u\right)}{u^2 + \tilde{v}^2}$$
$$= \gamma_+ g(v) + \gamma_- f(u)\,,$$

And $\tilde{v} = |\, v \,|$ is a real positive variable (while v is purely imaginary)

$$J_{\rm LV} = I_{\rm LV} - \tilde{c}\, H_{\rm LV}$$
$$= (x p_y + y p_x)^2 + \frac{\tilde{c}}{2}\left(p_x^2 + p_y^2\right) + \mathcal{U}$$

Where

$$\mathcal{U}(u,v) = \frac{\left(2\,u^2 + \tilde{c}\right) g(v) + \left(2\,\tilde{v}^2 - \tilde{c}\right) f(u)}{u^2 + \tilde{v}^2}$$
$$= \gamma_+ g(v) + \gamma_- f(u)\,,$$

One can then show that $0 < \gamma_\pm < 2$

Implying using (i) that $\begin{cases} \gamma_+ g(v) \geq -2|g_0| \\ \gamma_- f(u) \geq -2|f_0| \end{cases}$

and in turn that $\mathcal{U} \geq -2\left(|g_0| + |f_0|\right)$

(*i.e.* $\mathcal{U}$ is bounded below)

$$
\begin{aligned}
J_{\mathrm{LV}} &= I_{\mathrm{LV}} - \tilde{c}\, H_{\mathrm{LV}} \\
&= (x p_y + y p_x)^2 + \frac{\tilde{c}}{2}\left(p_x^2 + p_y^2\right) + \mathcal{U}
\end{aligned}
$$

Where

$$
\begin{aligned}
\mathcal{U}(u, v) &= \frac{\left(2\, u^2 + \tilde{c}\right) g(v) + \left(2\, \tilde{v}^2 - \tilde{c}\right) f(u)}{u^2 + \tilde{v}^2} \\
&= \gamma_+ g(v) + \gamma_- f(u)\,,
\end{aligned}
$$

As a consequence of J_{LV} being conserved and of $\mathcal{U}$ being bounded below

p_y, p_x and $\left| x\, p_x + y\, p_y \right|$ are bounded

Second step: show that x and y are bounded (separately)

$$J_{\mathrm{LV}} = I_{\mathrm{LV}} - \tilde{c}\, H_{\mathrm{LV}}$$

Where
$$= \left(x p_y + y p_x\right)^2 + \frac{\tilde{c}}{2}\left(p_x^2 + p_y^2\right) + \mathcal{U}$$

$$\mathcal{U}\left(u, v\right) = \frac{\left(2\, u^2 + \tilde{c}\right) g\left(v\right) + \left(2\, \tilde{v}^2 - \tilde{c}\right) f\left(u\right)}{u^2 + \tilde{v}^2}$$
$$= \gamma_+ g(v) + \gamma_- f(u)\,,$$

To that hand: we show that conditions (i) and (ii) implies that $\mathcal{U}$ grows without bounds at large enough $R = \sqrt{x^2 + y^2}$ which excludes that x and y grow without bound.

Steps are here a bit technical

The outcome is that, under conditions (i) and (ii) at least one of the following inequality holds

$$\mathcal{U} \geq -2\,|g_0| + F_0\ \ (\tilde{c}/2)^{\zeta/4}\ \ R^{\zeta/2}$$

$$\mathcal{U} \geq -2\,|f_0| + G_0\ \ (\tilde{c}/2)^{\eta/4}\ \ R^{\eta/2}$$

$$\mathcal{U} > -2\,|g_0| + \tilde{c}F_0 2^{-\frac{\zeta}{2}} \left(R\sqrt{2\tilde{c}}\right)^{\zeta/2-1}$$

$$\mathcal{U} \geq -2\,|f_0| + 4G_0 2^{-\frac{\eta}{2}} \left(R\sqrt{2\tilde{c}}\right)^{\frac{\eta}{2}}$$

$$\mathcal{U} \geq -2\,|g_0| + 4F_0 2^{-\frac{\zeta}{2}} \left(R\sqrt{2\tilde{c}}\right)^{2}$$

$$\mathcal{U} > -2\,|f_0| + \tilde{c}G_0 2^{-\frac{\eta}{2}} \left(R\sqrt{2\tilde{c}}\right)^{\eta/2-1}$$

This implies that for $F_0 > 0, G_0 > 0$ and $\eta > 2$, $\zeta > 2$ $\mathcal{U}$ diverges at large values of $R = \sqrt{x^2 + y^2}$.

NB: this proof also applies to the PP Liouville model

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V(x,y)$$

with

$$\begin{cases} V = \dfrac{f(u) - g(v)}{u^2 - v^2} \,, \\ I = -\,(xp_y - yp_x)^2 - c\,p_x^2 \\ \qquad + 2\,\dfrac{u^2 g(v) - v^2 f(u)}{u^2 - v^2} \,, \end{cases}$$

Where one « ghostifies x » via a complex

canonical transformation $\mathfrak{C}_x^{\pm}$ as opposed to $\mathfrak{C}_y^{\pm}$

(i) $f(u)$ and $g(v)$ are bounded below, i.e.,

$$f(u) \geqslant f_0$$
$$g(v) \geqslant g_0$$

with constants $f_0, g_0 \in \mathbb{R}$; and

(ii) at large $|u|$ and $|v|$, these lower bounds sharpen to

$$f(u) \geqslant 4F_0\,|u|^\zeta > 0$$
$$g(v) \geqslant 4G_0\,|v|^\eta > 0$$

with positive constants $F_0, G_0 \in \mathbb{R}^+$ as well as $\zeta > 2$ and $\eta > 2$

A set of functions f and g satisfying the above

Yields the theory considered in

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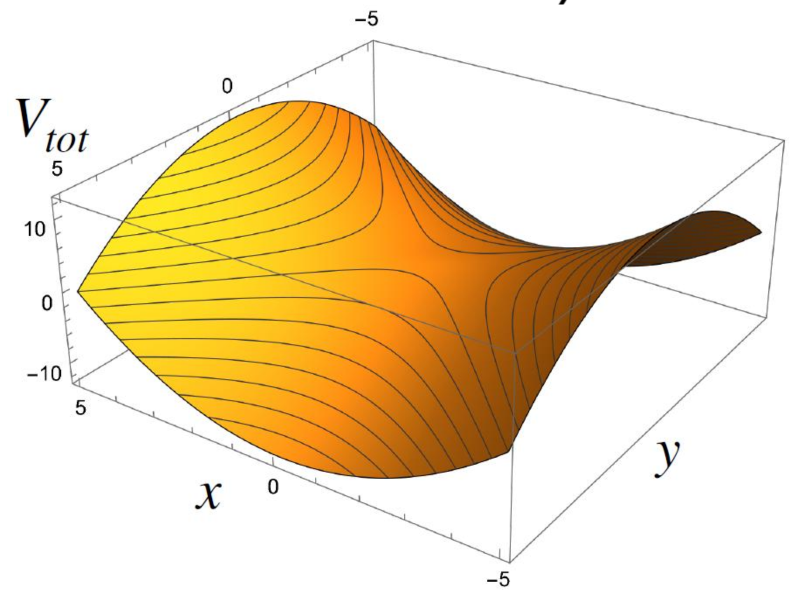
Is defined by the Hamiltonian

$$H = \frac{1}{2}\left(p_x^2 + x^2\right) - \frac{1}{2}\left(p_y^2 + y^2\right) + V_I\left(x, y\right)$$

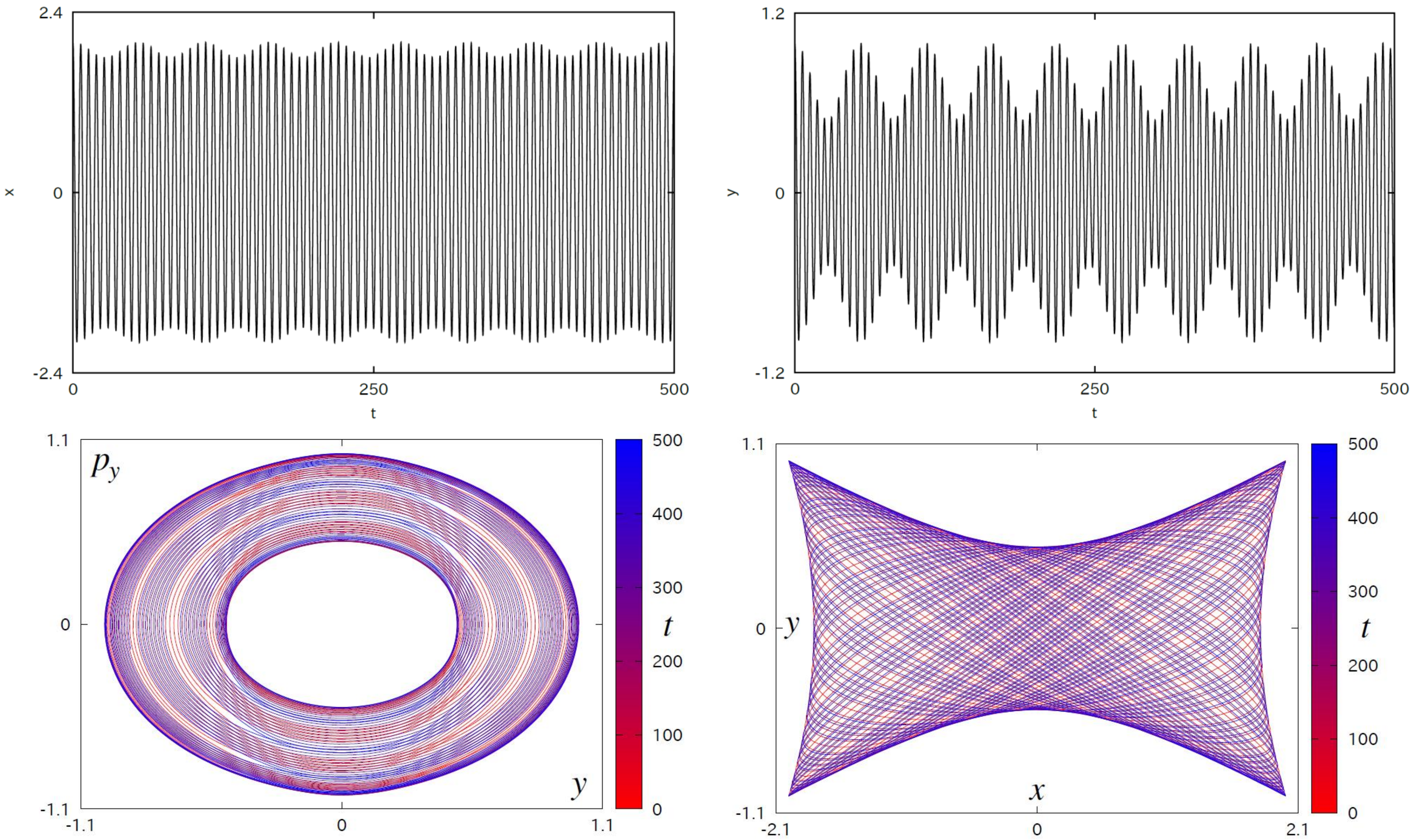
With $V_I\left(x, y\right) = \lambda\left(\left(x^2 - y^2 - 1\right)^2 + 4x^2\right)^{-1/2}$

and λ a constant



Total potential energy plotted here for the coupling constant $\lambda = \frac{1}{3}$

This yields indeed a totally stable motion in phase space despite the ghost and the unbounded above and below interaction potential



$$H = \frac{1}{2}\left(p_x^2 + x^2\right) - \frac{1}{2}\left(p_y^2 + y^2\right) + V_I(x, y)$$

$$V_I(x, y) = \lambda\left(\left(x^2 - y^2 - 1\right)^2 + 4x^2\right)^{-1/2}$$

Expanding the total potential energy around the origin, we find

$$V_{tot} = \frac{\omega_x^2}{2}x^2 - \frac{\omega_y^2}{2}y^2 + \lambda\left(x^4 + 4y^2x^2 + y^4\right) + \ldots$$

With $\begin{cases} \omega_x^2 = 1 - 2\lambda \\ \omega_y^2 = 1 + 2\lambda \end{cases}$

So for $|\,\lambda\,| < \frac{1}{2}$ both oscillators are linearly stable around the origin

Can one remove the interactions between the oscillators by making a suitable canonical transformation?

No !

This can be shown using a theorem by Arnold:

define $\begin{cases} z_x = p_x + ix \\ \bar{z}_x = p_x - ix \end{cases}$ … and similarly for y

then (in the « non resonant » case, where, as generically true, $\omega_x/\omega_y = \sqrt{(1-2\lambda)/(1+2\lambda)}$ is irrational) any interaction of the form $z_x^{\alpha_x}\,\bar{z}_x^{\beta_x}\,z_y^{\alpha_y}\,\bar{z}_y^{\beta_y}$ can be removed by a canonical transformation, except if $\alpha_x = \beta_x$ and simultaneously $\alpha_y = \beta_y$

In our case, looking e.g. at order 4,

each term x^4, x^2y^2 and y^4

contains one (and only one) monomial which cannot be

removed by a canonical transformation.

(i) $f(u)$ and $g(v)$ are bounded below, i.e.,

$$f(u) \geqslant f_0$$
$$g(v) \geqslant g_0$$

with constants $f_0, g_0 \in \mathbb{R}$; and

(ii) at large $|u|$ and $|v|$, these lower bounds sharpen to

$$f(u) \geqslant 4F_0\,|u|^\zeta > 0$$
$$g(v) \geqslant 4G_0\,|v|^\eta > 0$$

with positive constants $F_0, G_0 \in \mathbb{R}^+$ as well as $\zeta > 2$ and $\eta > 2$

The previous model is obtained from f and g (satisfying the above) and which are polynomial functions of u^2 and v^2

$$\begin{cases} f(u) = \mathcal{C}_0 + \mathcal{C}_1 u^2 + \mathcal{C}_2 u^4 \\ g(v) = \mathcal{D}_0 + \mathcal{D}_1 v^2 + \mathcal{D}_2 v^4 \end{cases} \quad \text{with} \quad \begin{cases} \mathcal{C}_2 > 0 \\ \mathcal{D}_2 > 0 \end{cases}$$

An other interesting set of polynomial functions f and g in the Liouville case are obtained by choosing identical f and g :

$$\begin{cases} f(u) = \sum_{n=1}^{N} \mathcal{C}_n \left(u^2\right)^n \\ g(v) = \sum_{n=1}^{N} \mathcal{C}_n \left(v^2\right)^n \end{cases}$$

This yields a potential $$V_{\mathrm{LV}}^{(N)} = \sum_{n=1}^{N} \frac{\mathcal{C}_n}{u^2 - v^2} \left[\left(u^2\right)^n - \left(v^2\right)^n\right]$$

which is polynomial in x and y

For N ≤ 4 and N ≤ 6 the potentials read respectively :

$$V_{\mathrm{LV}}^{(4)}(x,y) = \frac{\omega_x^2}{2}\left[x^2 - \frac{(x^2-y^2)^2}{c}\right] - \frac{\omega_y^2}{2}\left[y^2 - \frac{(x^2-y^2)^2}{c}\right] + \mathcal{C}_4\left[(x^2-y^2)^3 - c(x^4-y^4)\right]$$

$$\begin{aligned} V_{\mathrm{LV}}^{(6)}(x,y) = V_4(x,y) & \\ &+ \mathcal{C}_5(x^2-y^2)^4 \\ &+ c\,\mathcal{C}_5(x^2-y^2)^2(x^2-4y^2) \\ &+ c^2\mathcal{C}_5(3y^4-2x^4) \\ &+ \mathcal{C}_6(x^2-y^2)^5 \\ &+ c\,\mathcal{C}_6(x^2-y^2)^3(x^2-5y^2) \\ &+ c^2\mathcal{C}_6(x^2-y^2)(x^4-8x^2y^2+10y^4) \\ &+ c^3\mathcal{C}_6(6y^4-3x^4) \end{aligned}$$

With identical and polynomial functions $f(u)$ and $g(v)$, and at large values of $|u|$ and $|v|$,

we have
$$\begin{cases} f(u) \longrightarrow \mathcal{C}_N \left(u^2\right)^N \\ g(v) \longrightarrow (-1)^N \mathcal{C}_N \left(-v^2\right)^N \end{cases}$$

So that condition (ii)

(ii) at large $|u|$ and $|v|$, these lower bounds sharpen to

$$f(u) \geqslant 4F_0 \, |u|^\zeta > 0 \, ,$$
$$g(v) \geqslant 4G_0 \, |v|^\eta > 0 \, ,$$

with positive constants $F_0, G_0 \in \mathbb{R}^+$ as well as $\zeta > 2$ and $\eta > 2$

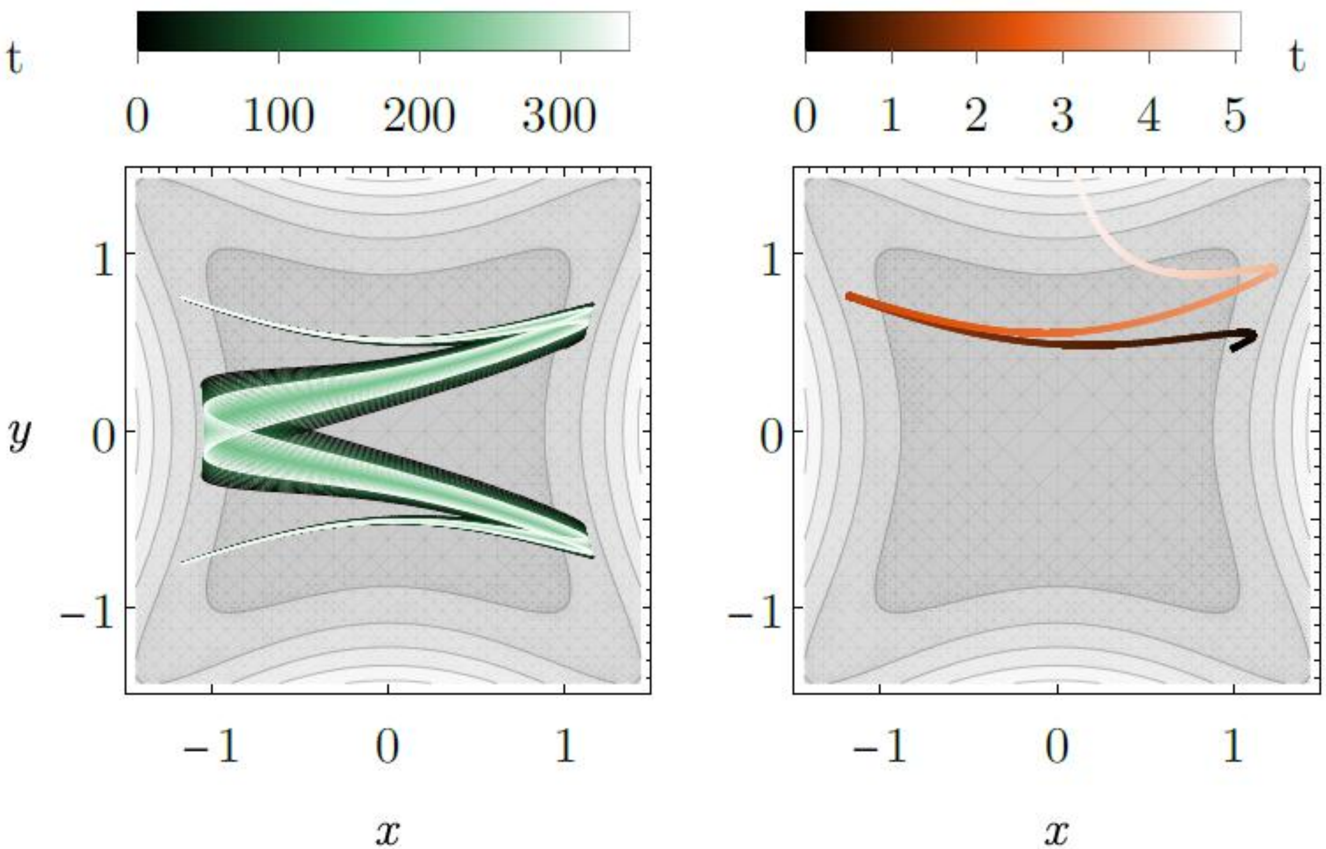
is fullfilled for **even N** but **not for odd N**

$$V_{\mathrm{LV}}^{(4)}(x,y) = \frac{\omega_x^2}{2}\left[x^2 - \frac{(x^2-y^2)^2}{c}\right] - \frac{\omega_y^2}{2}\left[y^2 - \frac{(x^2-y^2)^2}{c}\right] + \mathcal{C}_4\left[(x^2-y^2)^3 - c(x^4-y^4)\right] . \quad \text{(D7)}$$

Contrasting the motions for N = 3 (where $\mathcal{C}_4 = 0$) and N = 4

$$V_{\mathrm{LV}}^{(4)}(x,y) = \frac{\omega_x^2}{2}\left[x^2 - \frac{(x^2-y^2)^2}{c}\right] - \frac{\omega_y^2}{2}\left[y^2 - \frac{(x^2-y^2)^2}{c}\right] + \mathcal{C}_4\left[(x^2-y^2)^3 - c(x^4-y^4)\right] . \qquad \text{(D7)}$$

Two motions for N = 3 ($\omega_x^2 = 1,\ \omega_y^2 = -1,\ \mathcal{C}_4 = 0,\ \text{and}\ \tilde{c} = 1$)

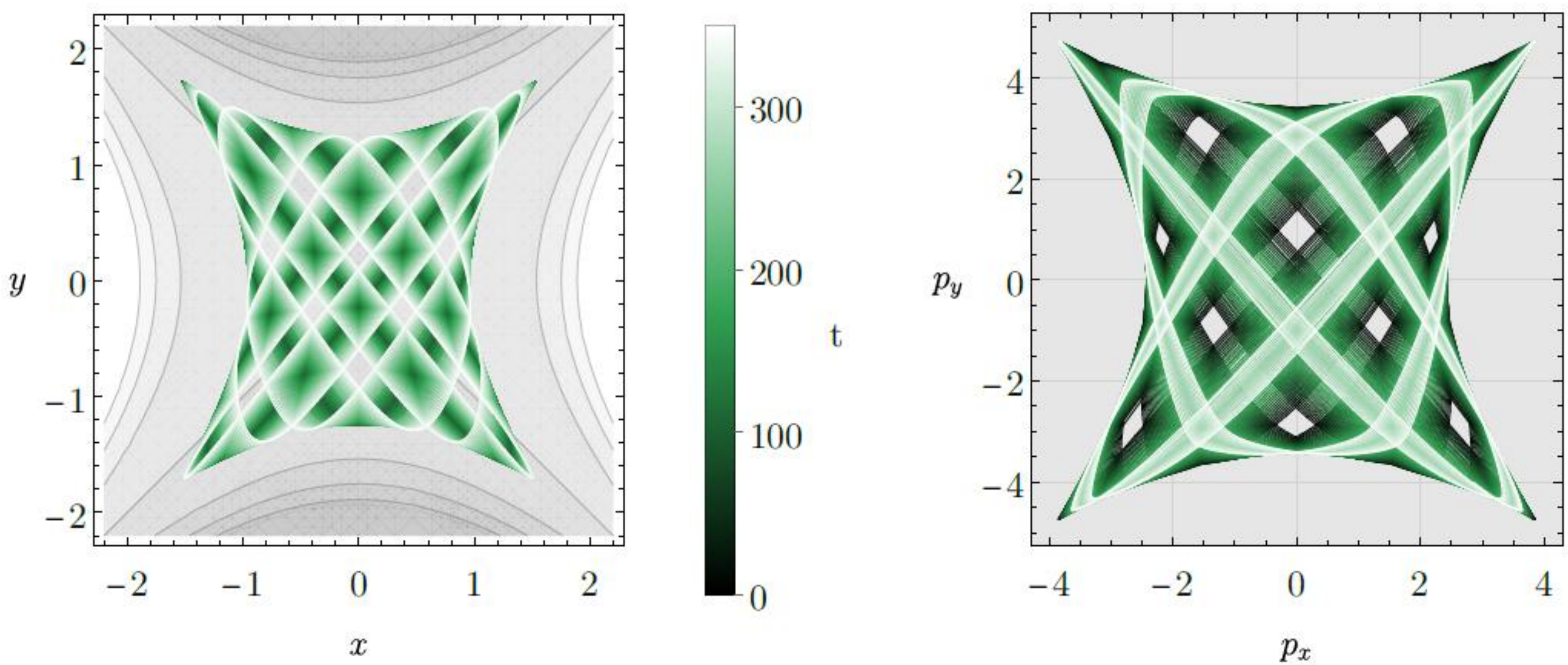


NB:

- In gray: contour plots of the potential
- One stable motion and a nearby diverging one

$$V_{\rm LV}^{(4)}(x,y) = \frac{\omega_x^2}{2}\left[x^2 - \frac{(x^2-y^2)^2}{c}\right] - \frac{\omega_y^2}{2}\left[y^2 - \frac{(x^2-y^2)^2}{c}\right] + \mathcal{C}_4\left[(x^2-y^2)^3 - c(x^4-y^4)\right] . \qquad \text{(D7)}$$

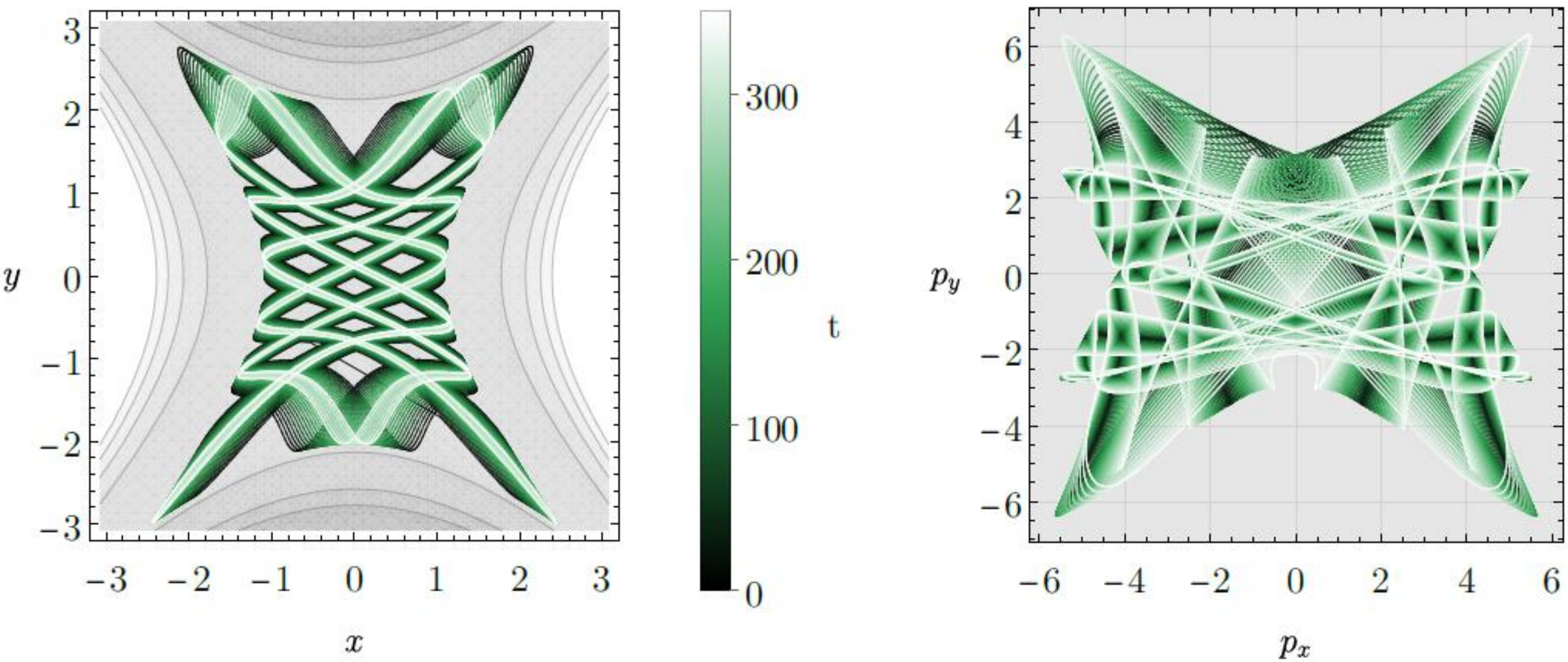
A motion for N = 4 ($\omega_x^2 = 1, \omega_y^2 = 1, \mathcal{C}_4 = 1$, and $\tilde{c} = 1$)



- In gray: contour plots of the potential
- origin ($x = 0, y = 0$) is non tachyonic

$$V_{\mathrm{LV}}^{(4)}(x,y) = \frac{\omega_x^2}{2}\left[x^2 - \frac{(x^2-y^2)^2}{c}\right] - \frac{\omega_y^2}{2}\left[y^2 - \frac{(x^2-y^2)^2}{c}\right] + \mathcal{C}_4\left[(x^2-y^2)^3 - c(x^4-y^4)\right] . \qquad \text{(D7)}$$

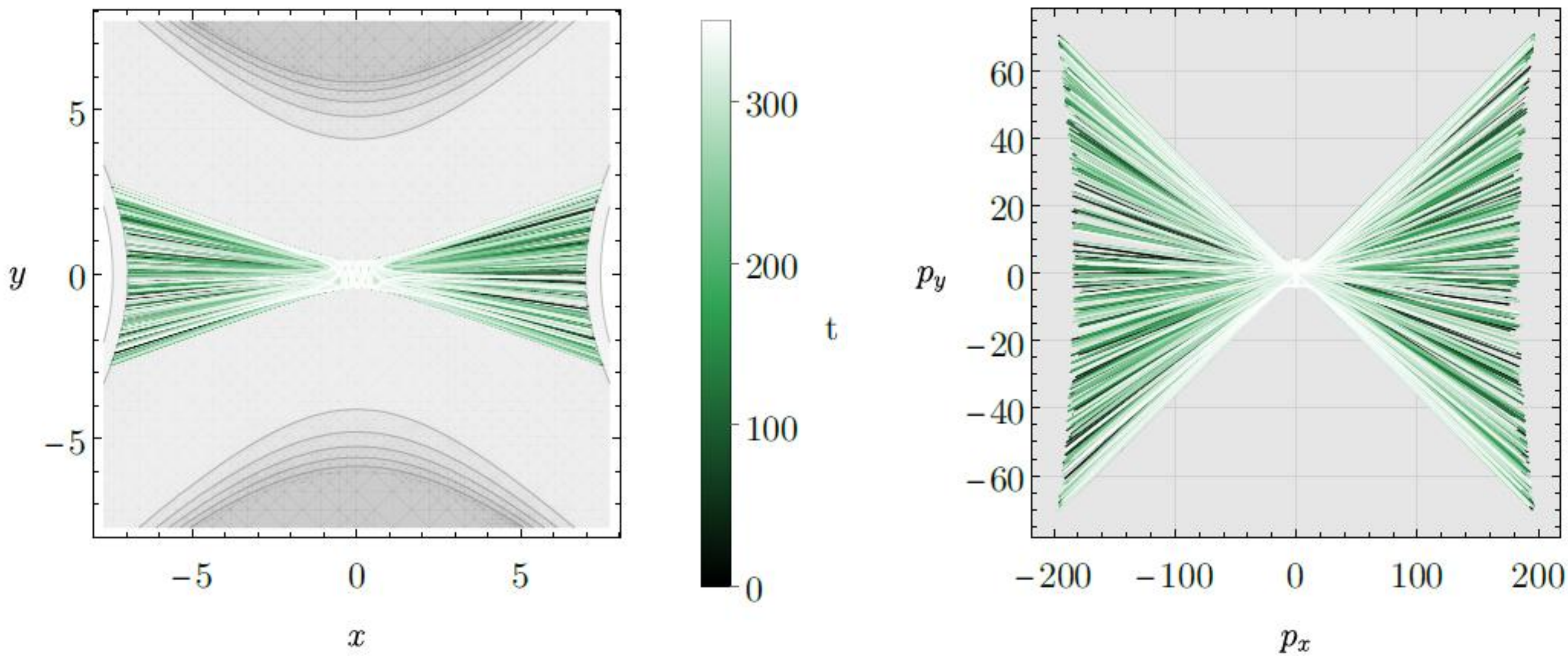
A motion for N = 4 ($\omega_x^2 = 5,\ \omega_y^2 = -5,\ = 1$, and $\tilde{c} = 1$)



- In gray: contour plots of the potential
- origin ($x = 0, y = 0$) is tachyonic

$$V_{\rm LV}^{(4)}(x,y) = \frac{\omega_x^2}{2}\left[x^2 - \frac{(x^2-y^2)^2}{c}\right] - \frac{\omega_y^2}{2}\left[y^2 - \frac{(x^2-y^2)^2}{c}\right] + \mathcal{C}_4\left[(x^2-y^2)^3 - c(x^4-y^4)\right] . \qquad \text{(D7)}$$

A motion for N = 4 ($\omega_x^2 = 1$, $\omega_y^2 = 100$, $\mathcal{C}_4 = 1$, and $\tilde{c} = 1$)



- In gray: contour plots of the potential
- Large frequency ratio $\frac{\omega_y}{\omega_x}$

V. Numerical investigations

Stability was also investigated numerically, both for integrable and non integrable models

Numerical method:

- Random initial conditions were drawn (from a normal distribution with standard deviation SD) for $x(t_\mathrm{i}), \mathrm{y}(\mathrm{t_i}), \dot{x}(t_i), \dot{y}(t_i)$ at $t_i = 0$

- These initial conditions were evolved until either $t_f = t_{max}$ or until the numerical routine **detects a runaway**, in which case $t_f < \mathrm{t}_{max}$

- We repeat the numerical experiment above N_{evol} times and define $t_\mathrm{mean} = \frac{1}{N_\mathrm{evol}} \sum_{j=1}^{N_\mathrm{evol}} t_f^{(j)}$

- A value of $t_{mean} = t_{max}$ indicates (choosing a large t_{max}) an absence of runaway

To detect a runaway we use

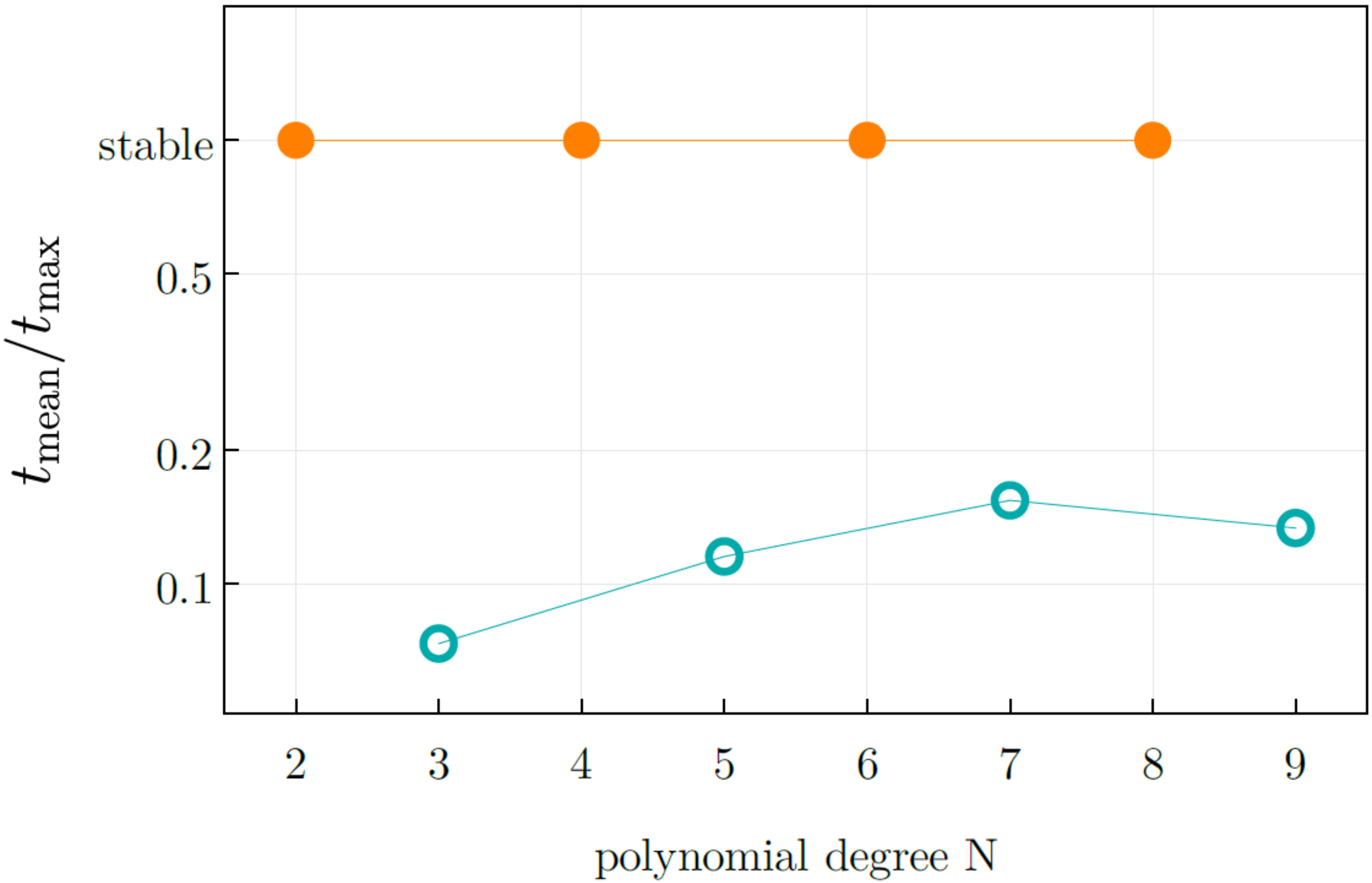
- <u>Criterion A:</u>

Stiffness detection by the the Runge-Kutta numerical routine (NDSolve of Mathematica). This detects typically (faster than) exponential runaways

- <u>Criterion B:</u>

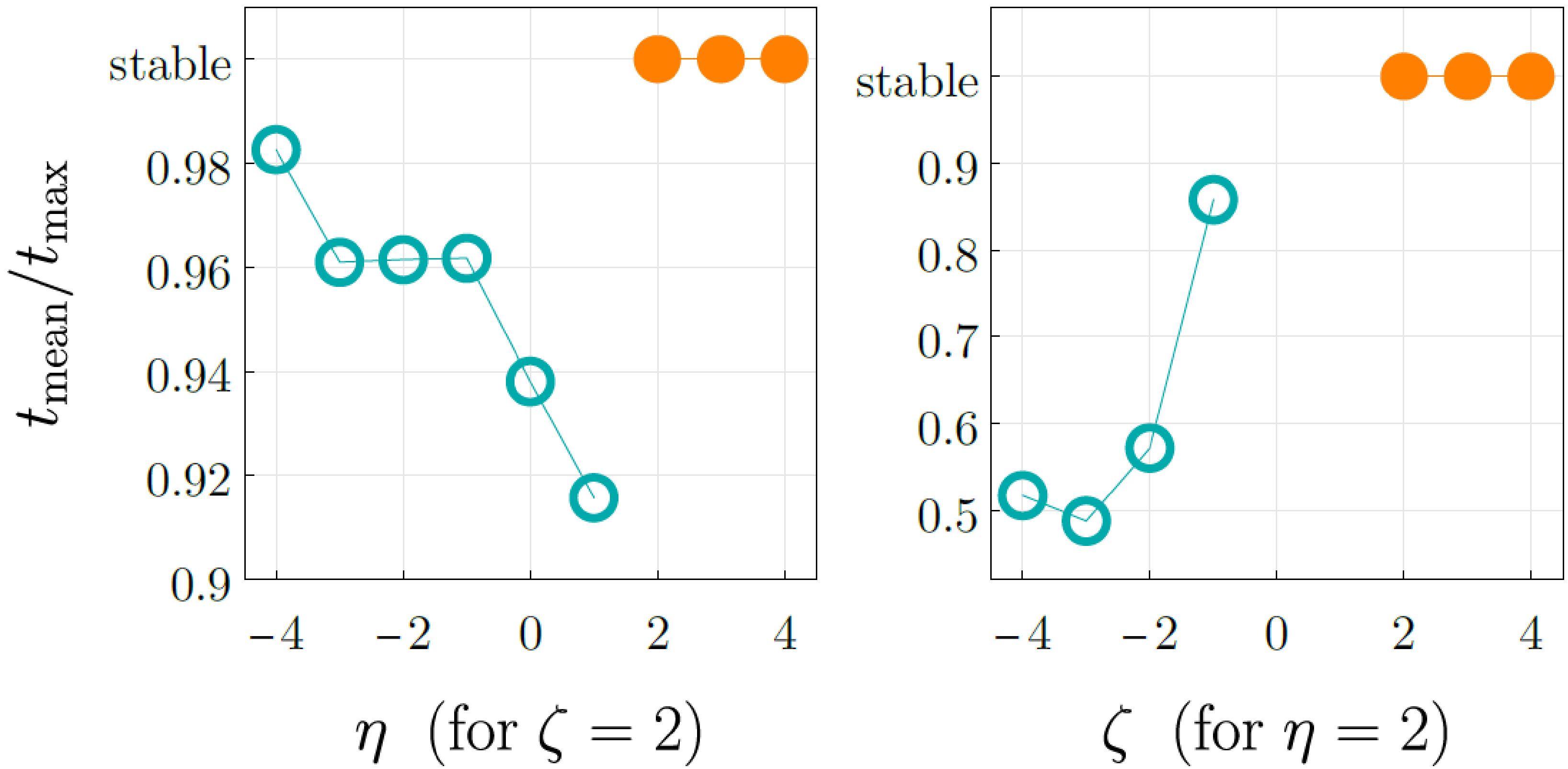
Polynomial growth are detected checking whenever $x(\mathrm{t})$ and $y(t)$ become larger than a specific constant chosen to be much larger than the standard deviation SD of the initial values.

Numerical investigations of the Liouville (integrable) model : agrees with the analytic proof



Going beyond the polynomial case, we also investigated

$$f(u) = |u|^{\zeta}\ , \quad g(v) = |v|^{\eta}$$



Agrees with criterion (ii) of the analytic proof

In the stable Liouville cases (i.e. even N) one notices the following features of the potentials:

- We can define the « decoupled » potentials for x and y respectively by

$$V_P(x) = V(x,0)$$
$$V_G(y) = V(0,y)$$

- And we have that $V(x,y) \lesseqgtr V_{\mathrm{P}}(x) + V_{\mathrm{G}}(y) \quad \forall\, |x| \gtrless |y|$

(note that attention has to been paid to what happens around $|x| = |y|$ direction)

For $N \leq 4$ we have

$$V_{\mathrm{LV}}^{(4)}(x,y) = \frac{\omega_x^2}{2}x^2 - \frac{\omega_y^2}{2}y^2 + \frac{1}{\tilde{c}}\left(\frac{\omega_x^2}{2} - \frac{\omega_y^2}{2}\right)(x^2 - y^2)^2 + \tilde{c}\,\mathcal{C}_4(x^4 - y^4) + \mathcal{C}_4(x^2 - y^2)^3$$

Both potentials leads to stable motions for the P and the G d.o.f. taken in isolation

$$V_{\mathrm{P}}(x) = \frac{\omega_x^2 x^2}{2} + \left[\frac{(\omega_x^2 - \omega_y^2)}{2\tilde{c}} + \mathcal{C}_4\tilde{c}\right]x^4 + \mathcal{C}_4 x^6$$

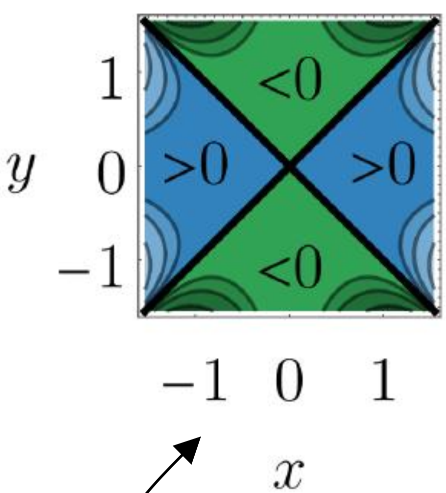
$$V_{\mathrm{G}}(y) = -\left[\frac{\omega_y^2 y^2}{2} + \left[\frac{(\omega_y^2 - \omega_x^2)}{2\tilde{c}} + \mathcal{C}_4\tilde{c}\right]y^4 + \mathcal{C}_4 y^6\right]$$

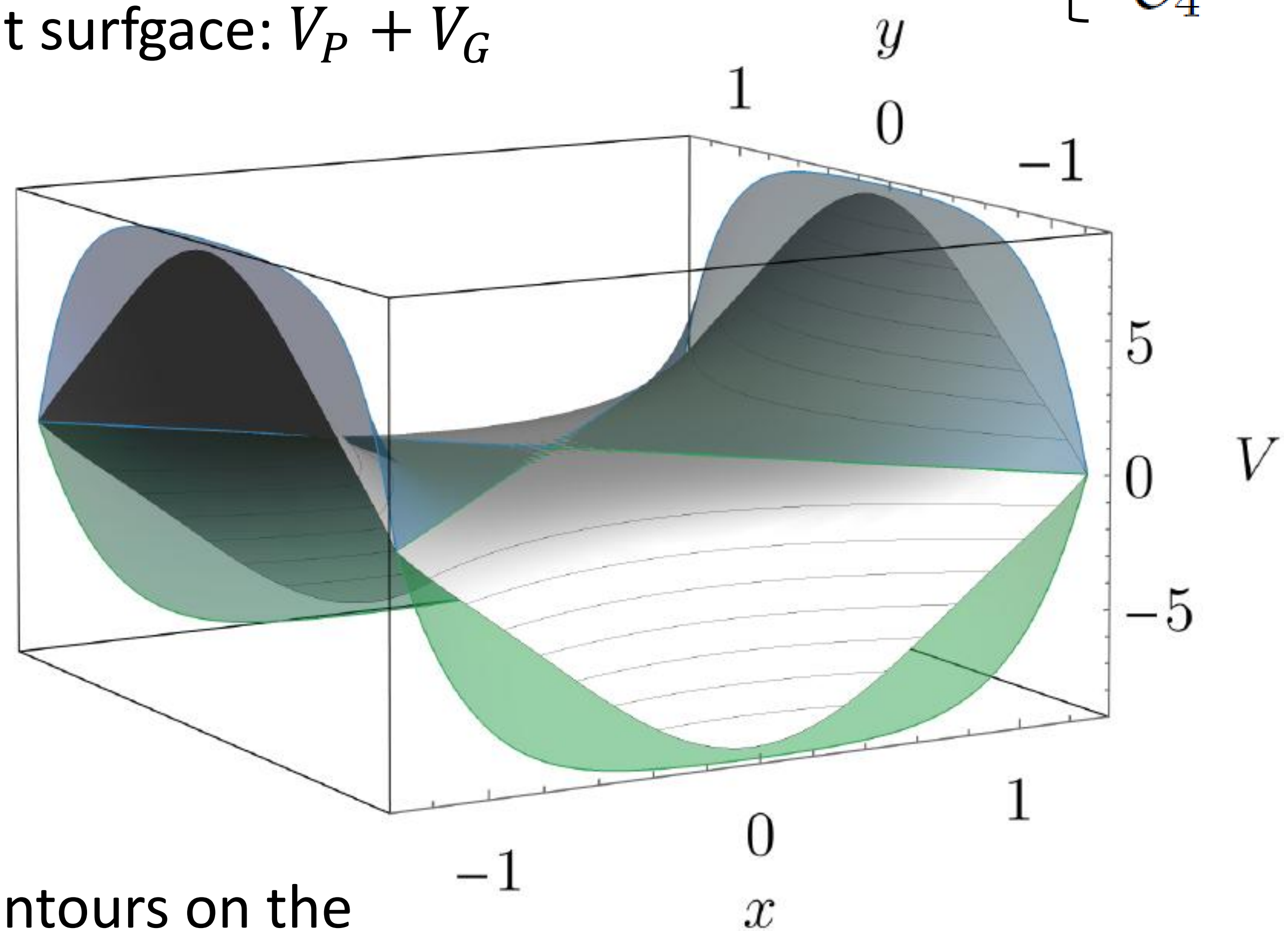
$$V(x,y) \lessgtr V_{\rm P}(x) + V_{\rm G}(y) \quad \forall\, |x| \gtrless |y|$$

V_{LV} @ N = 4 and $\begin{cases} \omega_x^2 = 1 \\ \omega_y^2 = 1 \\ \tilde{c} = 1 \\ \mathcal{C}_4 = 1 \end{cases}$

In green: V

As a transparent surfgace: $V_P + V_G$





Plots (as well a contours on the right) the difference $(V_P + V_G) - V$

From the integrable model,

We conjectured that **stable motion can exist** provided the **interaction between the P and the G is dominated at large x and y by decoupled stabilizing potentials**

i.e. **potentials which do not couple x and y but lead to stable motions of the P and G taken in isolation**

Note that this applies in the Liouville case (class 1)

$$V_{\rm LV}^{(4)}(x,y) = \frac{\omega_x^2}{2}x^2 - \frac{\omega_y^2}{2}y^2 + \frac{1}{\tilde{c}}\left(\frac{\omega_x^2}{2} - \frac{\omega_y^2}{2}\right)(x^2-y^2)^2 + \tilde{c}\,\mathcal{C}_4(x^4-y^4) + \mathcal{C}_4(x^2-y^2)^3$$

But not to e.g. a ghostified class 3 theory

$$V_{(\text{class }3)}^{(4)}(x,y) \overset{\omega_x=\omega_y}{=} \frac{\omega^2}{2}\left(x^2-y^2\right) + \mathcal{C}_4\left(x^2-y^2\right)^3 - \mathcal{C}_4\,c\,(x+y)(x-y)(x\pm y)^2\ .$$

For which we do not have analytical stabilty proof and found polynomial runaways in particular along $x = \pm y$ directions

Note also that similar polynomial runaways were also found for the theory (Smilga, 2017; Smilga, Roberts, 2008)

$$H = \frac{1}{2}\left(p_x^2 + \omega^2 x^2\right) - \frac{1}{2}\left(p_y^2 + \omega^2 y^2\right) + \frac{\lambda}{4\omega}(x-y)(x+y)^3\,,$$

which can in fact be shown to be a special ghostified class 8, but can also be obtained from a supersymmetric construction (Smilga, Roberts, 2008), and in fact our method to ghostify allows to get a host of such theories.

Consider in turn

$$V_{I,\delta} = \widetilde{V}_{I,\delta} + \frac{1}{2}\,x^2 - \frac{1}{2}\,y^2$$
$$\widetilde{V}_{I,\delta} = \left(1 + x^2 + y^2\right)^{1-\delta}$$

$$V_{II,\delta} = \widetilde{V}_{II,\delta} + \frac{1}{2}\,x^2 - \frac{1}{2}\,y^2$$
$$\widetilde{V}_{II,\delta} = \left(1 + x^2 y^2\right)^{\frac{1}{2}-\delta}$$

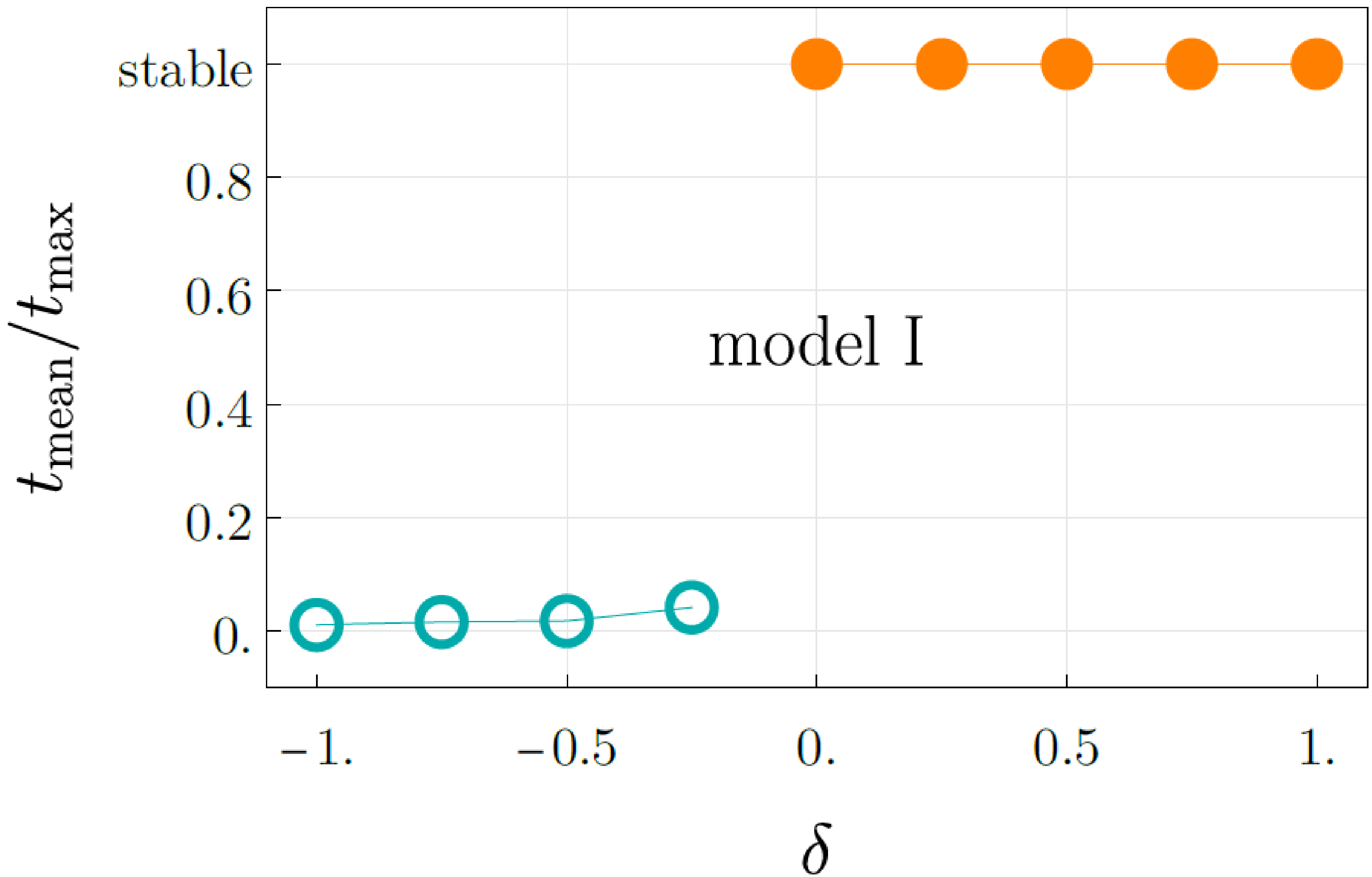
$$V_{III,\delta} = \widetilde{V}_{III,\delta} + \left(x^2\right)^{2+\delta} - \left(y^2\right)^{2+\delta}$$
$$\widetilde{V}_{III,\delta} = x^2\,y^2$$

All these theories are such that at large $|x|$ and $|y|$, and $\delta > 0$, the potential is dominated by stabilizing decoupled potentials for the two degrees of freedom.

Numerical checks :

$$V_{I,\delta} = \widetilde{V}_{I,\delta} + \frac{1}{2}\,x^2 - \frac{1}{2}\,y^2$$

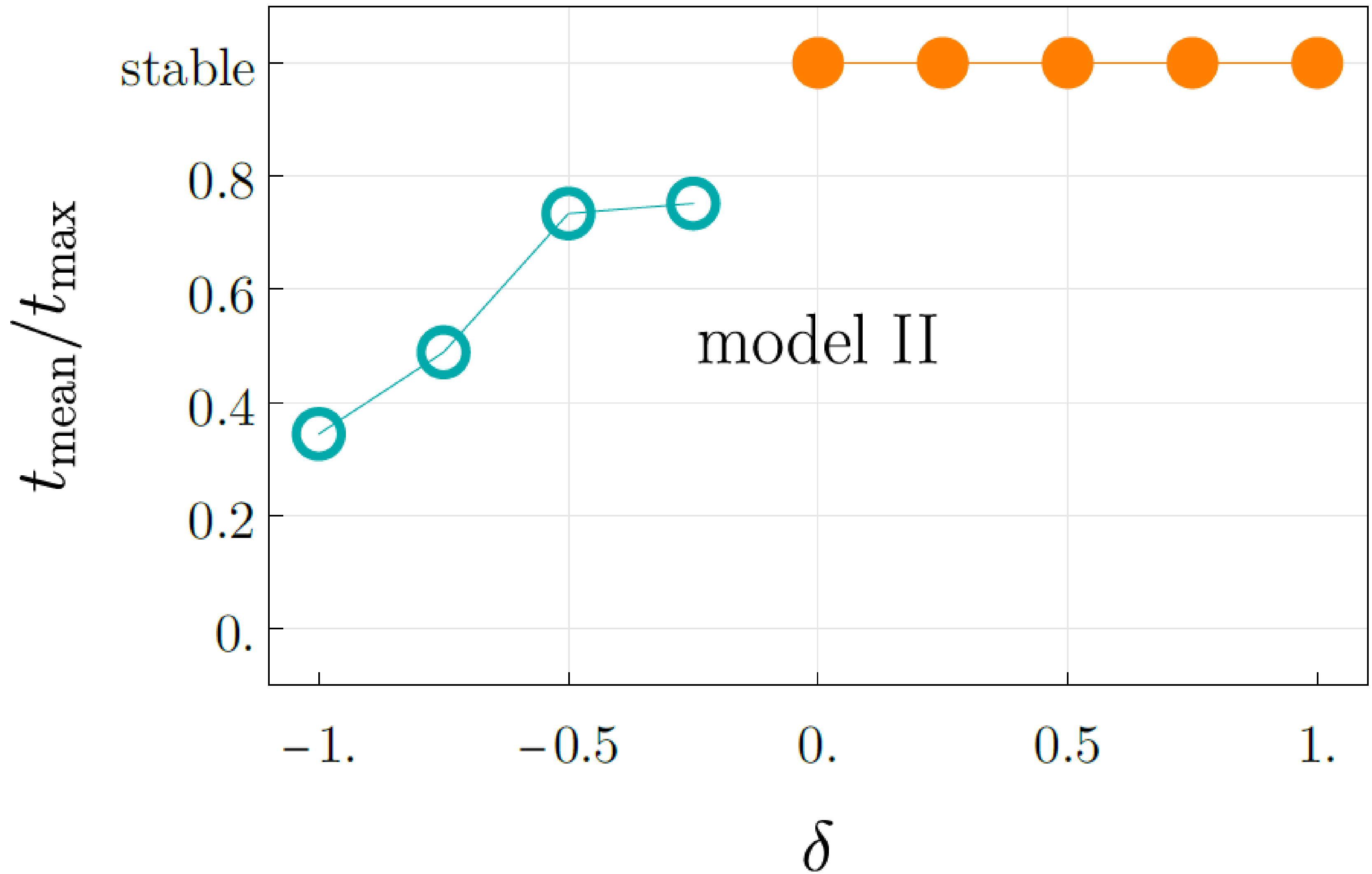
$$\widetilde{V}_{I,\delta} = \left(1 + x^2 + y^2\right)^{1-\delta}$$



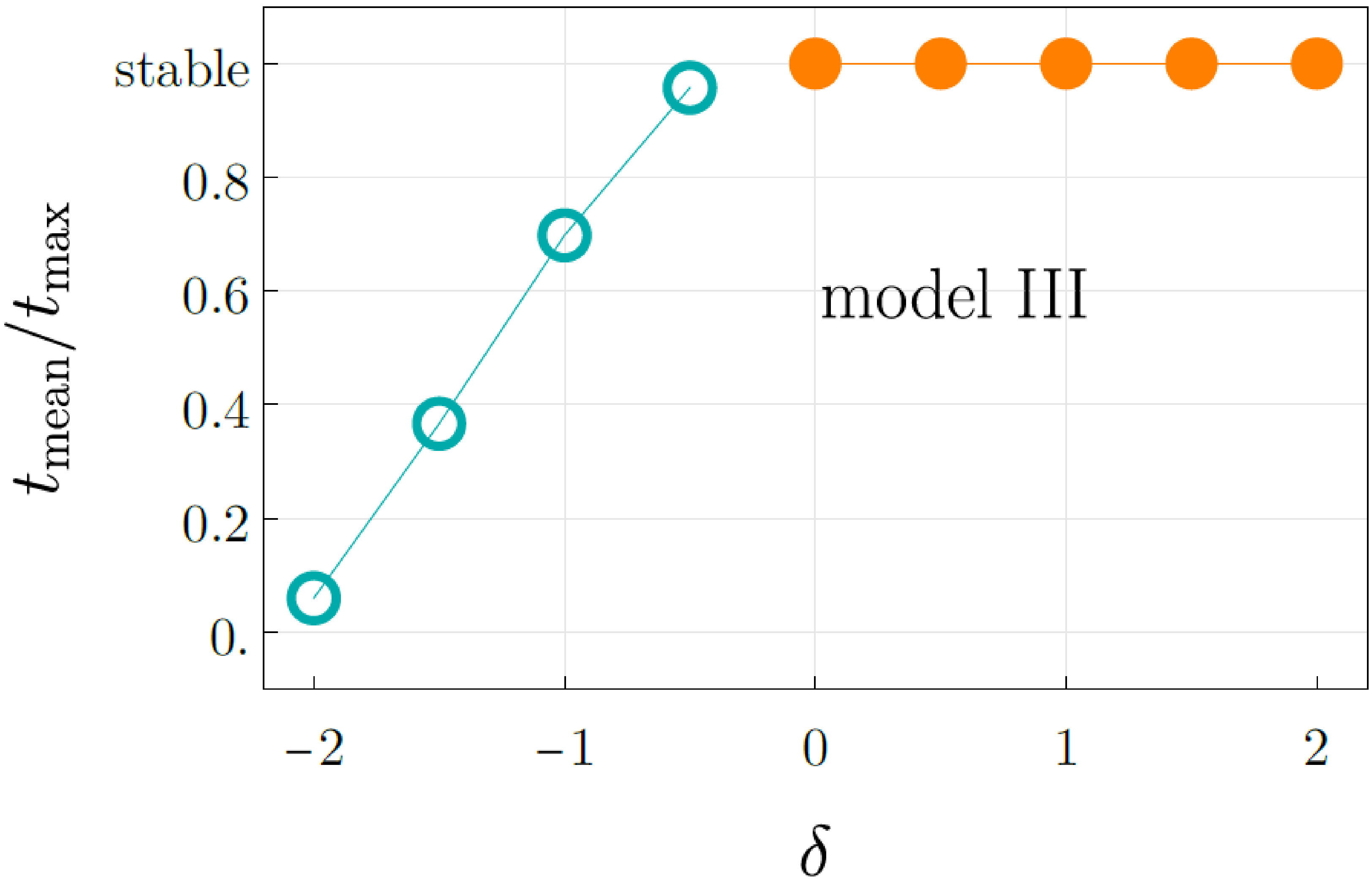
Numerical checks :

$$V_{II,\delta} = \widetilde{V}_{II,\delta} + \frac{1}{2}\,x^2 - \frac{1}{2}\,y^2$$

$$\widetilde{V}_{II,\delta} = \left(1 + x^2 y^2\right)^{\frac{1}{2}-\delta}$$



Numerical checks : $$V_{III,\delta} = \widetilde{V}_{III,\delta} + \left(x^2\right)^{2+\delta} - \left(y^2\right)^{2+\delta}$$
$$\widetilde{V}_{III,\delta} = x^2\, y^2$$



VI. Conclusions

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See Aaron's talk

Thank you for your attention !