# Adiabatic radial perturbations of relativistic stars: new solutions to an old problem

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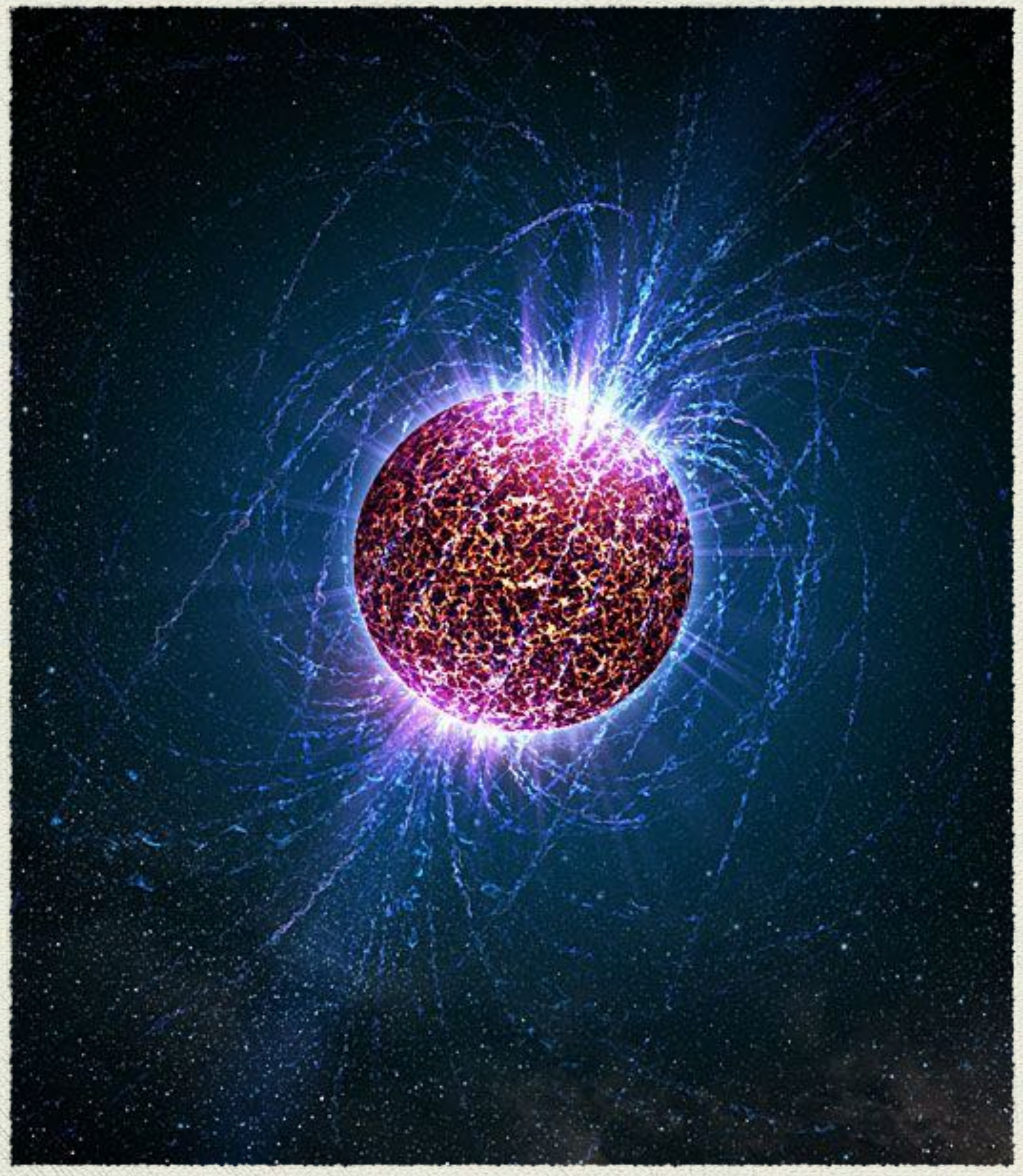
# Outline

- Stability of Relativistic stars and the Chandrasekhar pulsation equation
- Relativistic stars and LRS spacetimes
- Covariant description of LRS spacetimes
- Covariant gauge invariant adiabatic radial perturbations of relativistic stars
- Tales of two frames: Comoving and Static (with a Sturm-Liuville digression)
- Analytical and numerical solutions
- Some insight into non-equilibrium relativistic thermodynamics

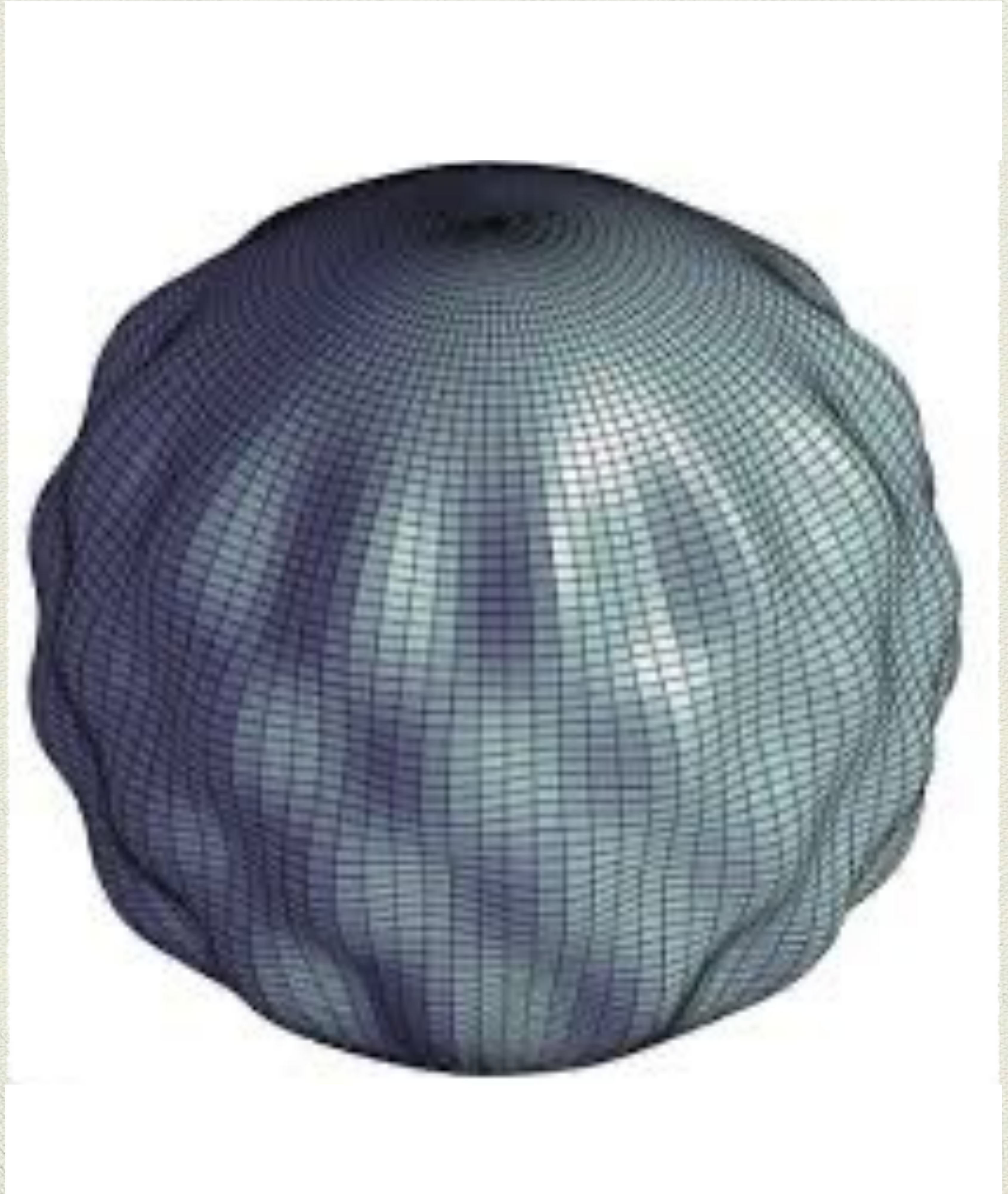
# Relativistic stars

- Relativistic Stars (RS) represent  a class of matter distributions with very high masses and small sizes such that their gravitational field  is "strong'''
- RSS includes dead stars, such as neutron stars and pulsars, and more exotic objects, e.g., boson stars.
- One of the most complex problems in  relativistic astrophysics is connected with the study of the perturbations of these objects
- The equations describing these perturbations were written by Chandrasekhar and collaborators



# Chandrasekar Pulsation Equation

- The perturbation for relativistic stars presents several physical and mathematical complications.
- However, in the case of adiabatic radial perturbations, they can be considerably simplified.
- In particular, those equations can be used to deduce general criteria for the stability of these objects.
- Chandrasekhar, in 1964, was the first to derive these equations for the specific case of radial perturbations



# Chandrasekar Pulsation Equation

The Chandrasekhar pulsation equation reads

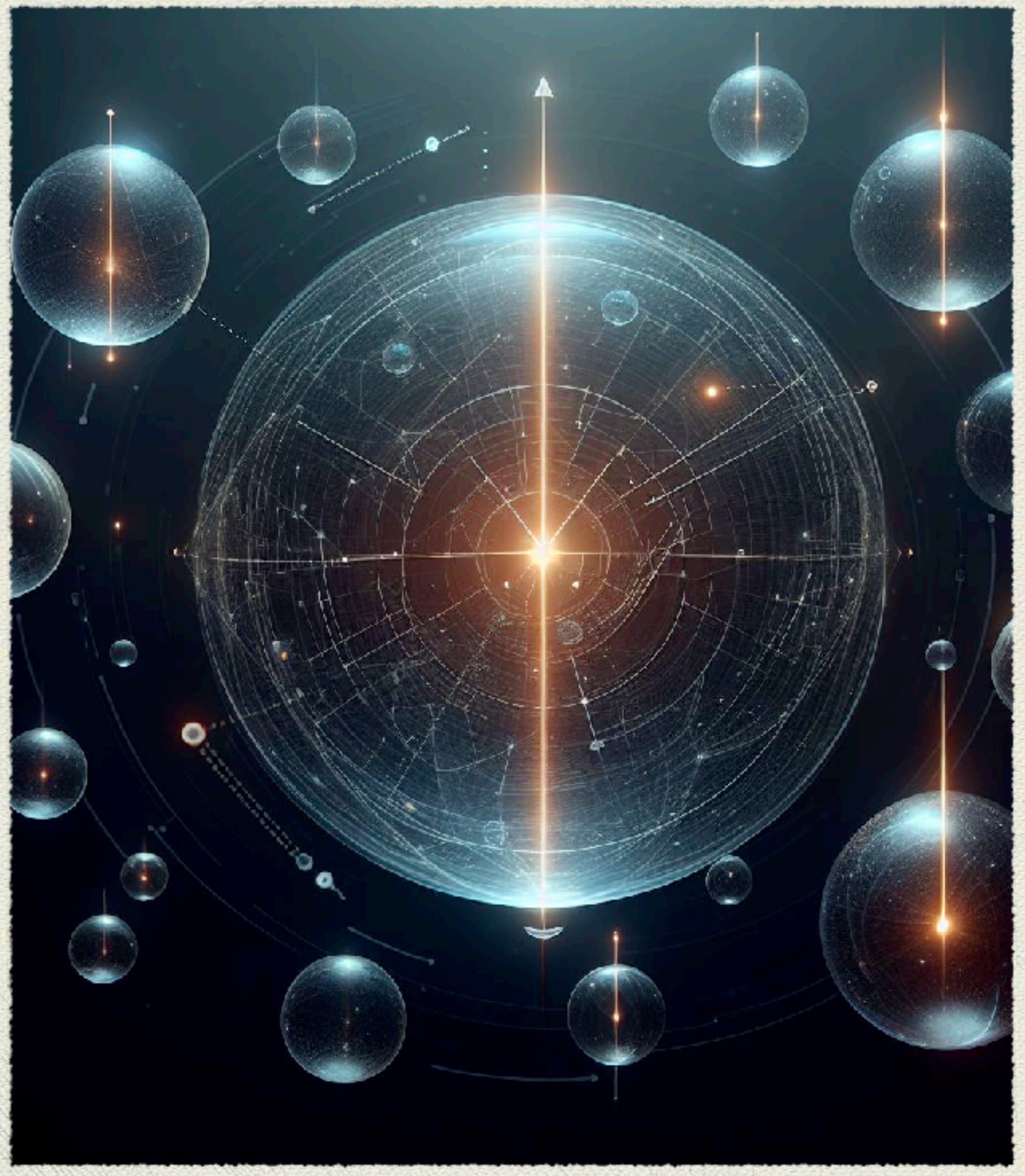
$$\sigma^2 e^{\lambda_0-\nu_0}(p_0+\epsilon_0)\,\xi = \frac{4}{r}\frac{dp_0}{dr}\,\xi - e^{-(\lambda_0+2\nu_0)/2}\frac{d}{dr}\left[e^{(\lambda_0+3\nu_0)/2}\frac{\gamma p_0}{r^2}\frac{d}{dr}\left(r^2 e^{-\nu_0/2}\xi\right)\right]$$
$$+\frac{8\pi G}{c^4}e^{\lambda_0}p_0(p_0+\epsilon_0)\,\xi - \frac{1}{p_0+\epsilon_0}\left(\frac{dp_0}{dr}\right)^2\xi\,.$$

- Describes the radial adiabatic perturbation of a self-gravitating sphere of fluid as described by an observer comoving with the matter flow
- Written in a gauge chosen in such a way that the coordinate of the background and the perturbed spacetime so that the gauge parameter $\xi$ encodes the perturbations
- This equation has the structure of a strum Liuville problem
- It gives information on the stability, i.e. on $\sigma$, of a given equilibrium via ansatz on $\xi$

Many extensions and refinements of these original equations were proposed…

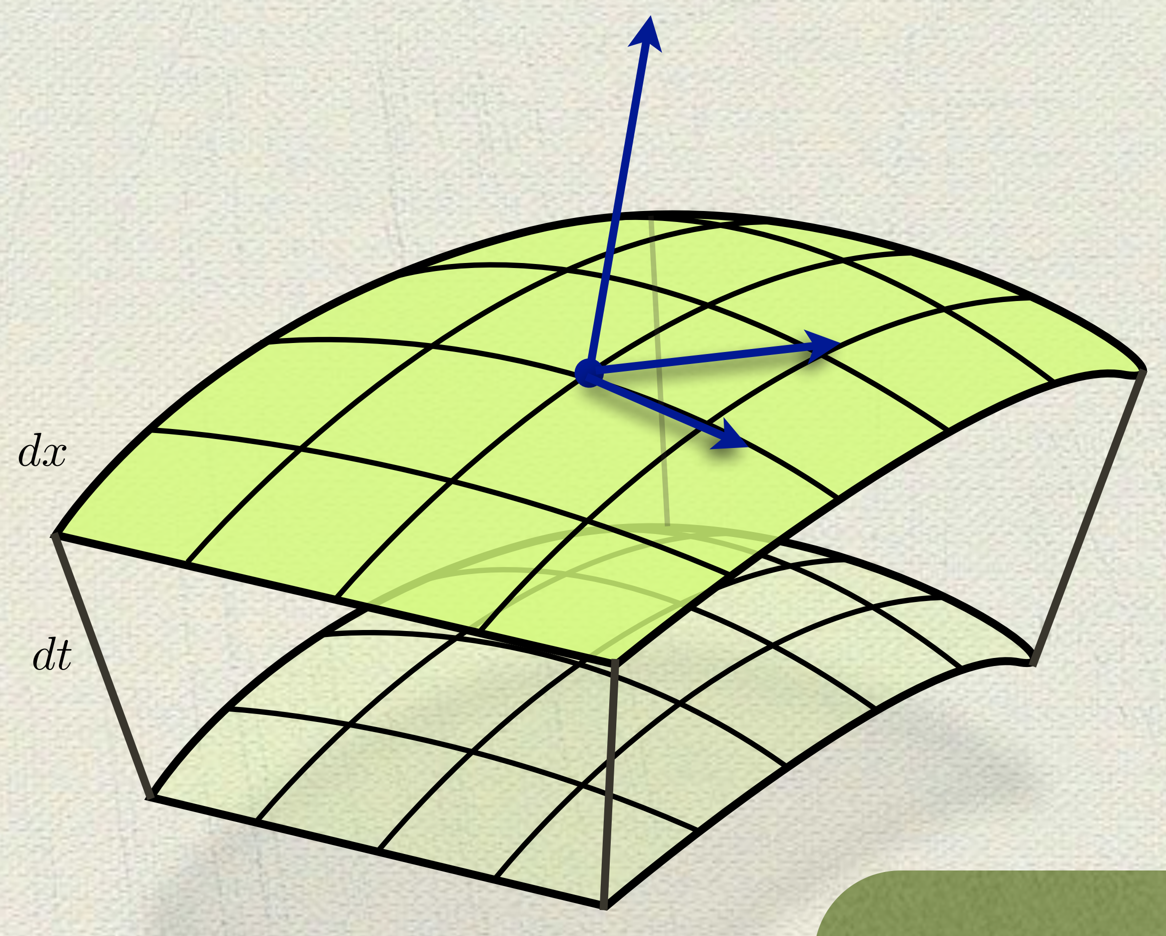
# Locally Rotationally Symmetric spacetimes

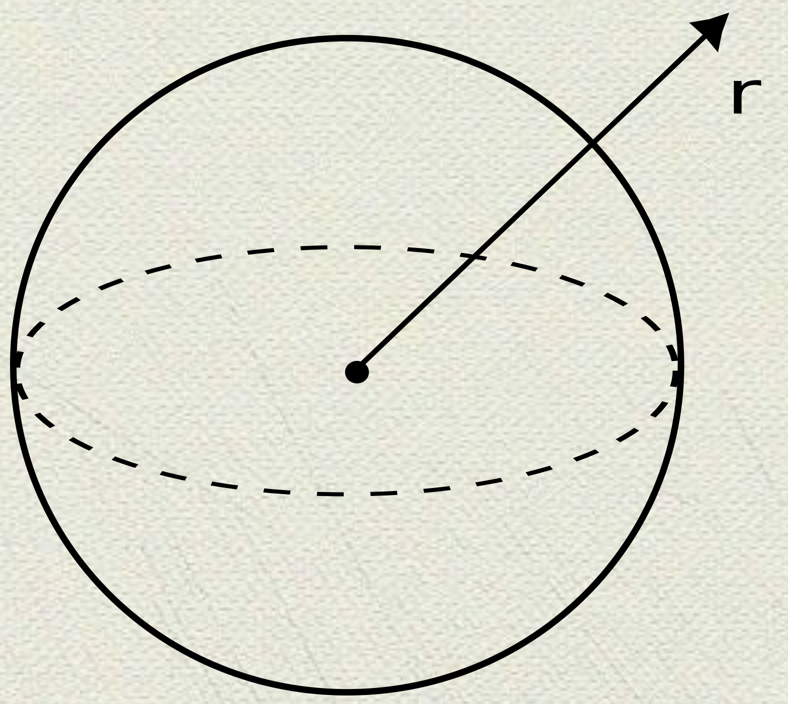
- LRS spacetimes are spacetimes characterized by the presence of local axes of symmetry in every point
- They describe a variety of interesting cosmological and astrophysical spacetimes (e.g., Goedel Universes, Bianchi models, Schwarzschild black holes, etc.)
- They can also be employed to model the spacetime associated with self-gravitating fluid distributions in and out of equilibrium.
- More specifically, the non-vortical subset of LRS spacetimes called LRS-II is suitable for describing non-rotating (and slowly rotating) relativistic stars



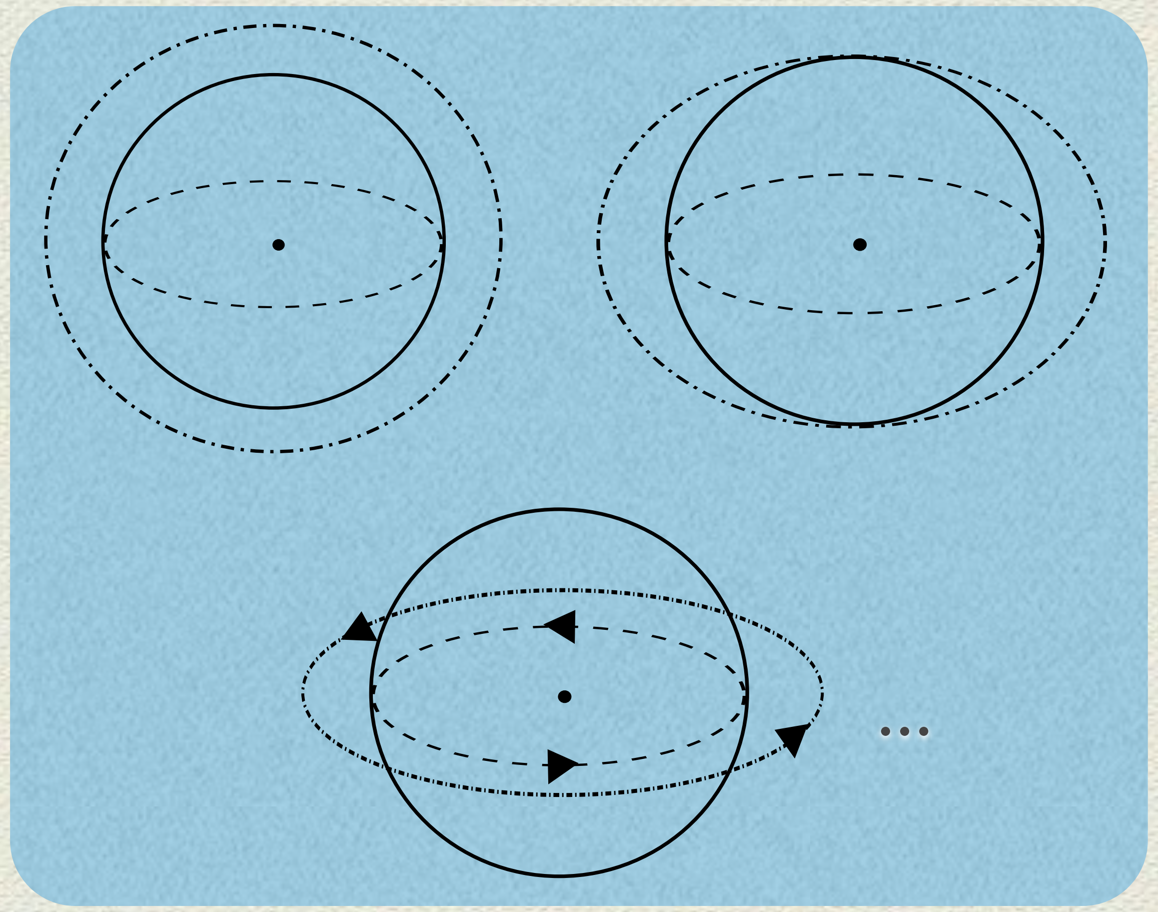
# 1+1+2 Descriprion of LRS spacetimes

A very effective tool to describe LRS spacetimes is the 1+1+2 covariant approach.




e.g. in the case of spherical symmetry




Thermodynamics

# 1+1+2 Descriprion of LRS spacetimes

- We single out a timelike vector $u_a$ and spacelike vector $e_a$.

$$g_{ab} = -u_a u_b + e_a e_b + N_{ab} = -u_a u_b + h_{ab}$$

- The projected derivative is split accordingly:

$$\dot{X}^{a..b}{}_{c..d} \equiv u^e \nabla_e X^{a..b}{}_{c..d}, \quad D_e X^{a..b}{}_{c..d} \equiv h^a{}_f \dots h^b{}_g h^p{}_c \dots h^q{}_d h^r{}_e \nabla_r X^{f..g}{}_{p..q},$$

$$\hat{X}_{a..b}{}^{c..d} \equiv e^f D_f X_{a..b}{}^{c..d}, \quad \delta_e X_{a..b}{}^{c..d} \equiv N_a{}^f \dots N_b{}^g N_i{}^c \dots N_j{}^d N_e{}^p D_p X_{f..g}{}^{i..j}.$$

# 1+1+2 Descriprion of LRS spacecetimes

- Kinematics of the vectors and the decomposition of the Weyl tensor give geometry of the spatial 2-hypersurfaces. For example

$$\nabla_a u_b = -\mathscr{A} u_a e_b + \left(\frac{1}{3}\theta + \Sigma\right) e_a e_b + \left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right) N_{ab} + \Omega \epsilon_{ab} ,$$

- The stress-energy tensor can be written in the irreducible form

$$T_{ab} = \mu\, u_a u_b + p_e\, e_a e_b + p_N\, N_{ab} + 2Q\, u_{(a} e_{b)}$$

$$p_e = p + \Pi \qquad p_N = p - \frac{1}{2}\Pi$$

Exploiting the Ricci and Bianchi identities, we can then give a set of evolution, propagation, and constraint equations for these quantities…

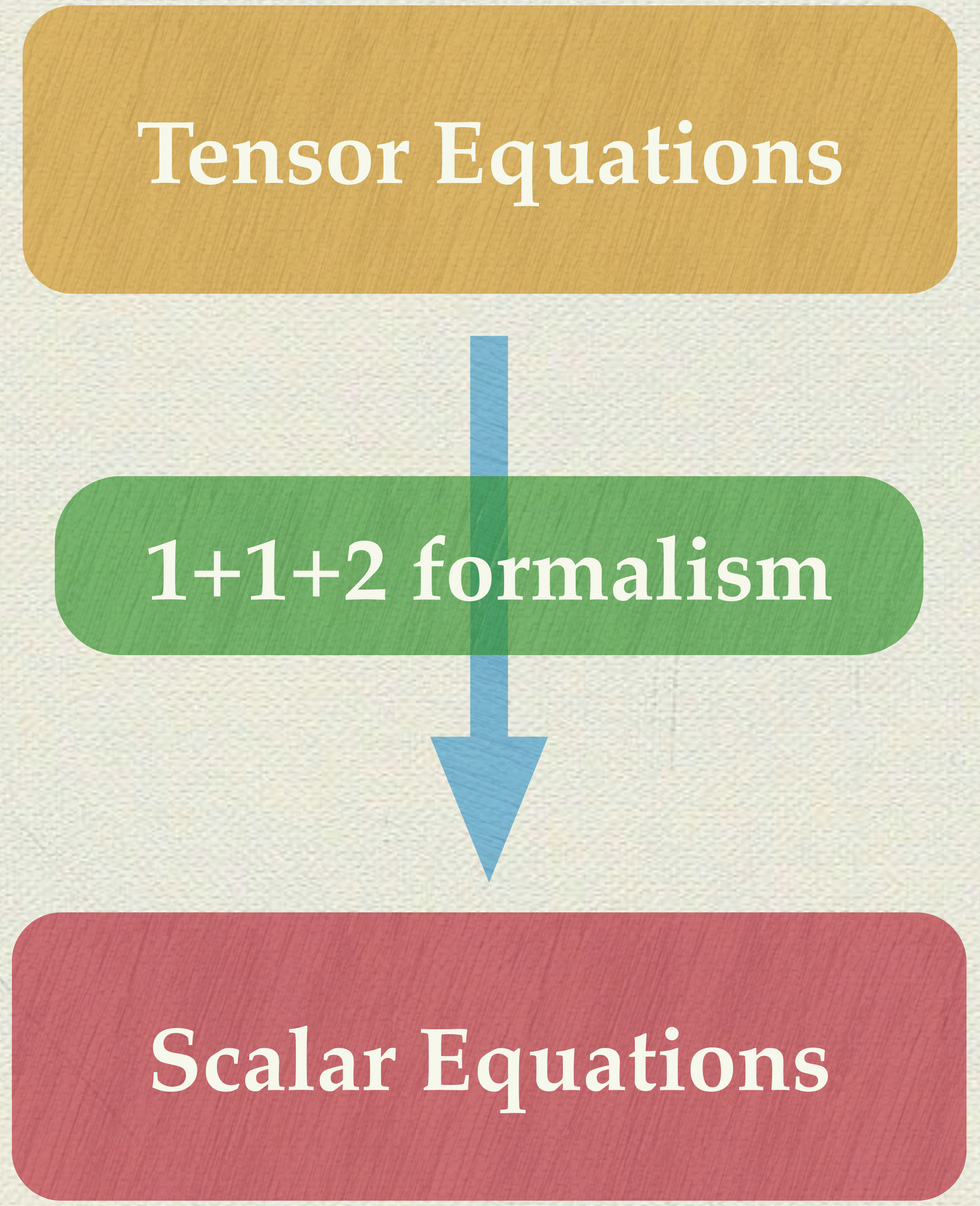
# 1+1+2 Descriprion of LRS spacetimes

$$\mathcal{A} = e_a \dot{u}^a,$$
$$\phi = \delta_a e^a,$$
$$\theta = D_a u^a,$$
$$\Sigma = \frac{1}{3} D_a u_b \left(2e^a e^b - N^{ab}\right)$$
$$\Omega = \frac{1}{2}\varepsilon^{ab}\delta_{[a}u_{b]} \quad ,$$
$$\xi = \frac{1}{2}\varepsilon^{ab}\delta_a e_b,$$
$$\mathcal{E} = C^{ab}{}_{cd} u_a u^d e_b e^c,$$
$$\mathcal{H} = \frac{1}{2}\varepsilon^{a}{}_{de} C^{deb}{}_{c} u^c e_a e_b,$$

Geometry

$$\mu = T_{ab} u^a u^b,$$
$$p = \frac{1}{3} T_{ab} \left(e^a e^b + N^{ab}\right),$$
$$\Pi = \frac{1}{3} T_{ab} \left(2e^a e^b - N^{ab}\right),$$
$$Q = -T_{ab} e^a u^b,$$

Thermodynamics



# LRS 1+1+2 Covariant Equations

$$\dot\phi + \left(\Sigma - \frac{2}{3}\theta\right)\left(\mathcal{A} - \frac{1}{2}\phi\right) - 2\xi\Omega = Q,$$

$$\dot\Sigma - \frac{2}{3}\dot\theta - \frac{1}{2}\left(\Sigma - \frac{2}{3}\theta\right)^2 + \mathcal{A}\phi + \mathcal{E} + 2\Omega^2 = = \frac{1}{3}(\mu + 3p) + \frac{1}{2}\Pi,$$

$$\dot{\mathcal{E}} - \frac{1}{3}\dot\mu + \frac{1}{2}\dot\Pi = \frac{3}{2}\left(\Sigma - \frac{2}{3}\theta\right)\mathcal{E} + \frac{1}{4}\left(\Sigma - \frac{2}{3}\theta\right)\Pi + 3\mathcal{H}\xi + \frac{1}{2}\phi Q - \frac{1}{2}(\mu + p)\left(\Sigma - \frac{2}{3}\theta\right),$$

$$\dot{\mathcal{H}} - \frac{3}{2}\left(\Sigma - \frac{2}{3}\theta\right)\mathcal{H} + 3\mathcal{E}\xi = Q\Omega + \frac{3}{2}\Pi\xi,$$

$$\dot\xi - 2\left(\Sigma - \frac{1}{6}\theta\right)\xi = 0,$$

$$\dot\Omega - \left(\Sigma - \frac{2}{3}\theta\right)\Omega - \mathcal{A}\xi = 0.$$

Evolution

$$\hat\phi + \frac{1}{2}\phi^2 - \left(\Sigma - \frac{2}{3}\theta\right)\left(\Sigma + \frac{1}{3}\theta\right) + \mathcal{E} - 2\xi^2 = -\frac{2}{3}\mu - \frac{1}{2}\Pi,$$

$$\hat\Sigma - \frac{2}{3}\hat\theta + \frac{3}{2}\phi\Sigma + 2\xi\Omega = -Q,$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat\mu + \frac{1}{2}\hat\Pi = 3\mathcal{H}\Omega + \frac{1}{2}\left(\Sigma - \frac{2}{3}\theta\right)Q - \frac{3}{2}\phi\mathcal{E} - \frac{3}{4}\phi\Pi,$$

$$\hat{\mathcal{H}} + \frac{3}{2}\phi\mathcal{H} + 3\mathcal{E}\Omega = -\left(\mu + p - \frac{1}{2}\Pi\right)\Omega - Q\xi,$$

$$\hat\xi + \phi\xi - \left(\Sigma + \frac{1}{3}\theta\right)\Omega = 0,$$

$$\hat\Omega - (\mathcal{A} - \phi)\Omega = 0.$$

Propagation

$$\hat{\mathcal{A}} + (\mathcal{A} + \phi)\mathcal{A} - \dot\theta - \frac{1}{3}\theta^2 - \frac{3}{2}\Sigma^2 + 2\Omega^2 = \frac{1}{2}(\mu + 3p),$$

$$\dot\mu + \hat Q + (2\mathcal{A} + \phi)Q + \theta(\mu + p) + \frac{3}{2}\Sigma\Pi = 0,$$

$$\dot Q + \hat p + \hat\Pi + \mathcal{A}(\mu + p + \Pi) + + \frac{3}{2}\phi\Pi + \left(\Sigma + \frac{4}{3}\theta\right)Q = 0.$$

Mixed

$$3\xi\Sigma - (2\mathcal{A} - \phi)\Omega - \mathcal{H} = 0.$$

$$3\phi\xi + \Omega(3\Sigma - 2\theta) = 0,$$

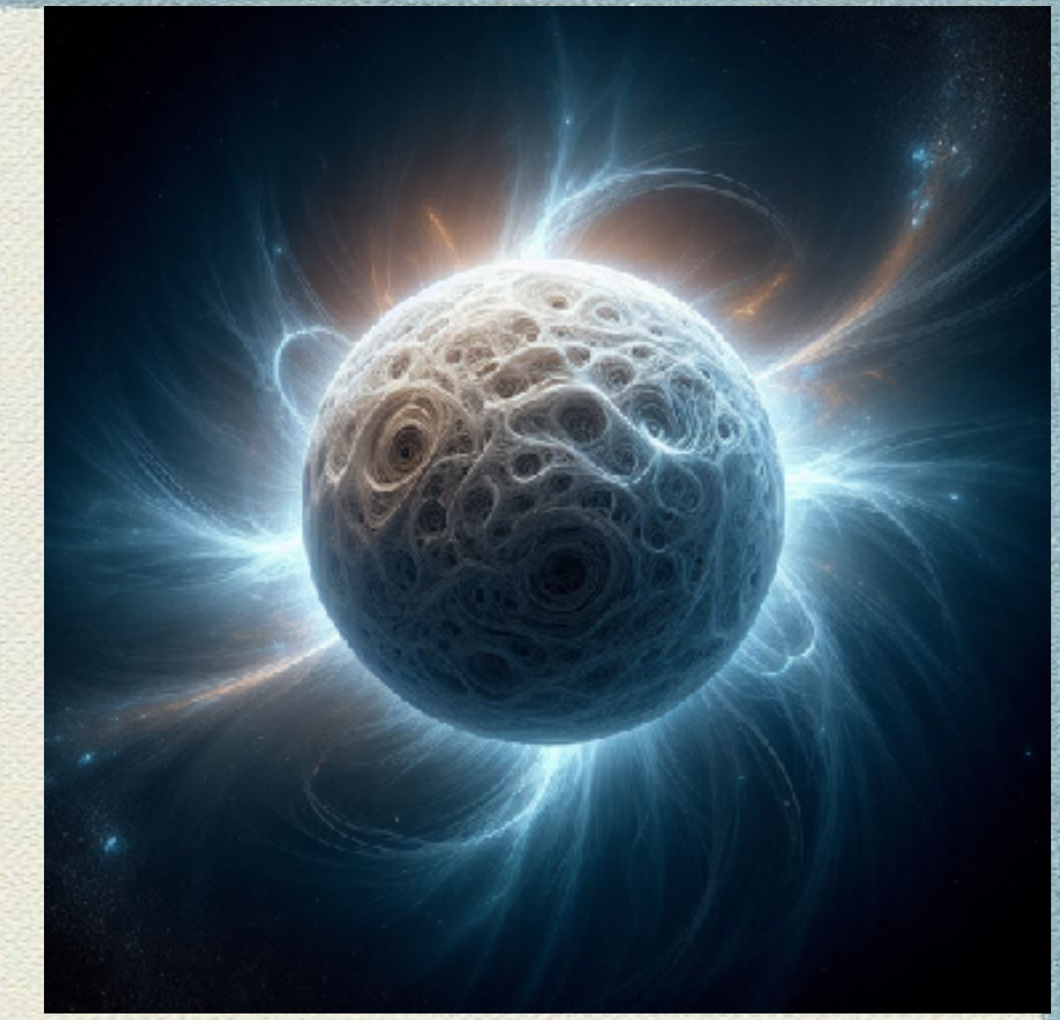
$$Q(\xi^2 + \Omega^2) - (p + \Pi + \mu)\xi\Omega = 0,$$

$$\frac{3}{2}\xi\Sigma\phi + \xi Q - 2\Omega^3 + \Omega\left[\frac{1}{2}\Pi + \frac{1}{3}\mu + p - \mathcal{E} + \xi^2 - \mathcal{A}\phi + \frac{1}{2}\left(\frac{2}{3}\theta - 3\Sigma\right)^2\right] = 0.$$

Constraints

# Background spacetime

- As a background, we consider two LRS-II spacetimes: one representing the vacuum exterior and the other the interior of a relativistic star
- As exterior, we consider the Schwarzschild solution
- As interior a non-vacuum solution of the equations for a non-rotating perfect fluid gaseous sphere (zero subscript = background quantity)



$$\phi_0 = \frac{2}{r\sqrt{(g_0)_{tt}}}\,,$$
$$\mathcal{A}_0 = \frac{1}{2\,(g_0)_{tt}\,\sqrt{(g_0)_{rr}}}\frac{d\,(g_0)_{tt}}{dr}\,.$$

$$\widehat{\mathcal{A}}_0 = \frac{1}{2}\left(\mu_0 + 3p_0\right) - \mathcal{A}_0\left(\mathcal{A}_0 + \phi_0\right)\,,$$
$$\widehat{\phi}_0 = -\frac{1}{2}\phi_0^2 - \frac{2}{3}\mu_0 - \mathcal{E}_0\,,$$
$$\widehat{p}_0 = -\left(\mu_0 + p_0\right)\mathcal{A}_0\,,$$
$$3\mathcal{E}_0 = \mu_0 + 3p_0 - 3\mathcal{A}_0\phi_0$$

- These solutions are joined by the standard Israel junction conditions

# Perturbations of LRS-II spacetimes

- The above formalism can be used to construct a covariant gauge invariant theory of perturbations
- We will limit ourselves here to describing this theory only in the case of LRS-II background and to radial, i.e., monopole perturbations.
- We will also assume that the perturbation does not produce anisotropic stresses or vorticity
- The first step is to choose a set of gauge invariant variables (Stewart-Walker). The most useful ones for our case are the following:

$$\mathsf{m} := \dot{\mu}\,,\ \mathsf{p} := \dot{p}\,,\ \mathsf{A} := \dot{\mathcal{A}}\,,\ \mathsf{F} := \dot{\phi}\,,\ \mathsf{E} := \dot{\mathcal{E}}$$

$$\theta := D_a u^a, \quad \Sigma := \frac{1}{3} D_a u_b \left(2e^a e^b - N^{ab}\right)$$

We will then use the 1+1+2 equation to derive evolution propagation and constraint equations for these variables

# Linearized Perturbations Equations

- We will consider higher order all terms that contain a non-linear combination of the perturbation variables: $\mathsf{mE} \Rightarrow$ Higher order $\quad \mathsf{m}p_0 \Rightarrow$ First order

$$\widehat{\mathsf{E}} - \frac{1}{3}\widehat{\mathsf{m}} = \left(\widehat{\mathscr{E}}_0 - \frac{1}{3}\hat{\mu}_0\right)\left(\frac{1}{3}\theta + \Sigma\right) + \frac{1}{3}\mathscr{A}_0\mathsf{m} - \frac{3}{2}\mathscr{E}_0\mathsf{F} - \left(\mathscr{A}_0 + \frac{3}{2}\phi_0\right)\mathsf{E}, \qquad \frac{2}{3}\hat{\theta} - \widehat{\Sigma} = Q + \frac{3}{2}\phi_0\Sigma,$$

$$\widehat{\mathsf{F}} = \hat{\phi}_0\left(\frac{1}{3}\theta + \Sigma\right) - (\mathscr{A}_0 + \phi_0)\,\mathsf{F} - \frac{2}{3}\mathsf{m} - \mathsf{E}, \qquad \widehat{Q} + \mathsf{m} = -\left(\phi_0 + 2\mathscr{A}_0\right)Q - \left(\mu_0 + p_0\right)\theta,$$

$$\widehat{\mathsf{A}} - \ddot{\theta} = \frac{1}{2}\left(\mathsf{m} + 3\mathsf{p}\right) + \widehat{\mathscr{A}}_0\left(\frac{1}{3}\theta + \Sigma\right) - \left(3\mathscr{A}_0 + \phi_0\right)\mathsf{A} - \mathscr{A}_0\mathsf{F}$$

$$\ddot{Q} + \widehat{\mathsf{p}} = \widehat{p}_0\left(\frac{1}{3}\theta + \Sigma\right) - \mathscr{A}_0\left(\mathsf{m} + 2\mathsf{p}\right) - \left(\mu_0 + p_0\right)\mathsf{A}$$

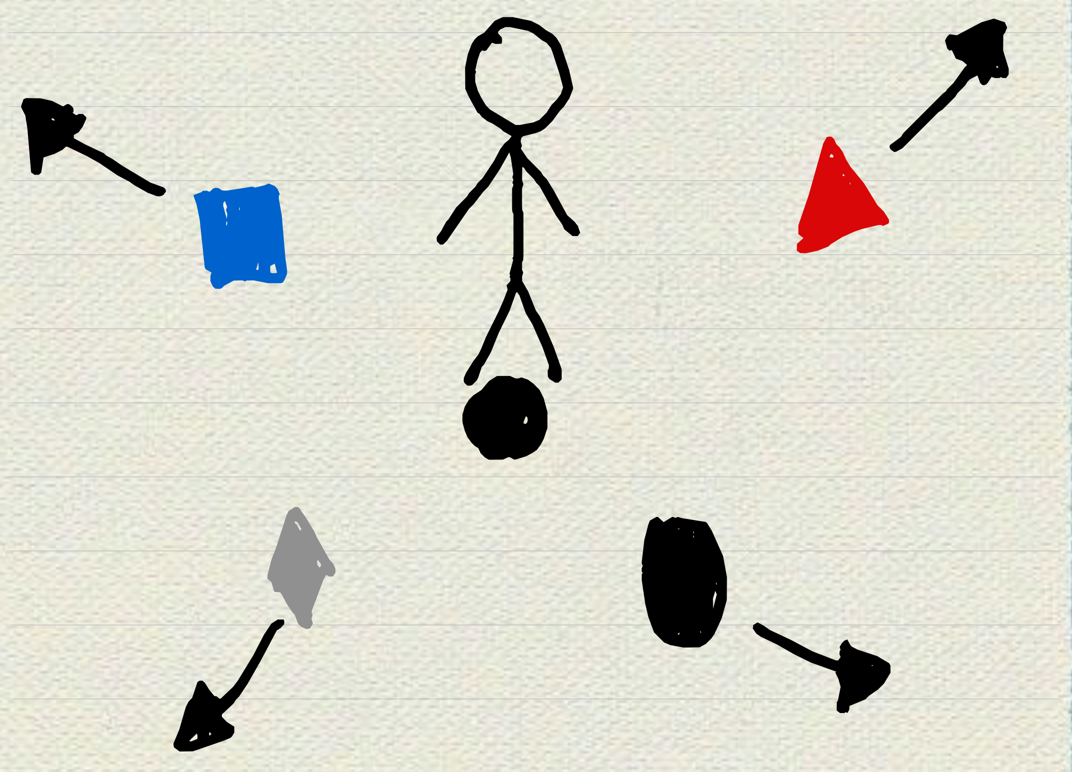
$$\mathsf{F} = Q + \left(2\mathscr{A}_0 - \phi_0\right)\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right),$$

$$\mathsf{E} = \left(\frac{1}{2}\phi_0 - \frac{4}{3}\mathscr{A}_0\right)Q - \left(2p_0 + \mathscr{A}_0\phi_0\right)\left(\frac{1}{3}\theta - \frac{1}{2}\Sigma\right) - \mathsf{m}$$

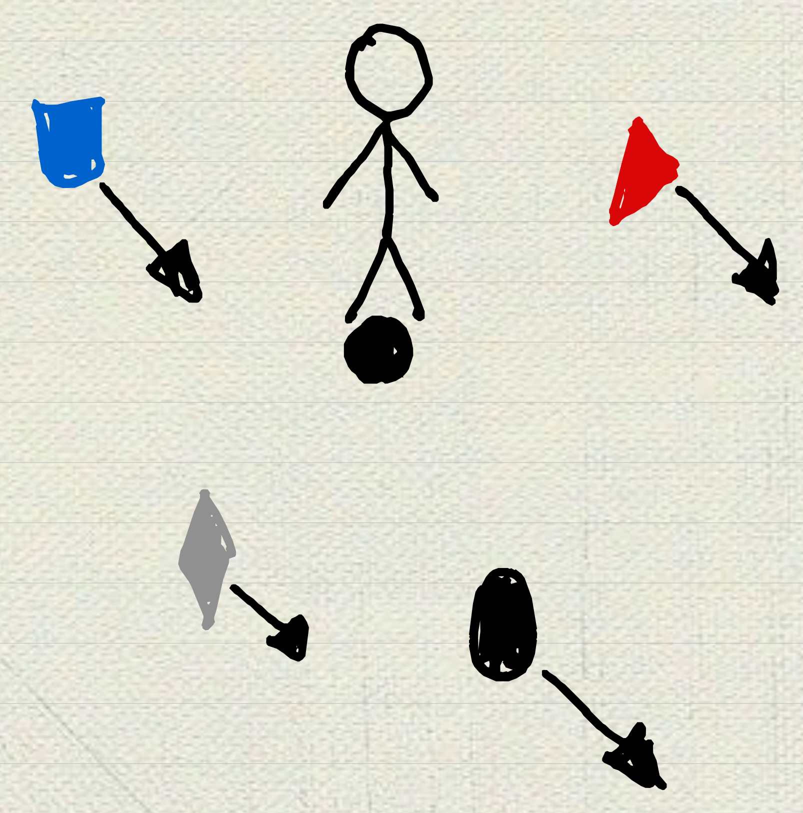
$$\ddot{\Sigma} - \frac{2}{3}\ddot{\theta} = \frac{1}{3}\left(\mathsf{m} + 3\mathsf{p}\right) - \mathsf{E} - \mathscr{A}_0\mathsf{F} - \phi_0\mathsf{A},$$

# Comoving and Static Frames

- The equations above can be written in any frame.
- Two frames are especially relevant: the comoving frame and the static frame.
- The comoving frame is a Lagrangian frame, i.e., it represents an observer that is at rest with respect to the fluid source.



- In terms of the 1+1+2 variables, the comoving frame is characterized by $Q = 0$.
- The static frame is an Eulerian frame, i.e., it represents an observer that is at rest with respect to the center of the fluid distribution.



- In terms of the 1+1+2 variables, the static frame is characterized by $3\theta - 2\Sigma = 0$, but $Q \neq 0$

# Change of frame

- Let us consider two observers one characterized by $(u, e)$, the other by $(\bar{u}, \bar{e})$ connected by the general relation (generalized Lorentz transformation)

$$\bar{u}^\alpha = u^\alpha \cosh\beta + e^\alpha \sinh\beta\,,$$
$$\bar{e}^\alpha = u^\alpha \sinh\beta + e^\alpha \cosh\beta\,.$$

- In the barred system, the 1+1+2 quantities transform as

$$\bar{\mu} = \mu - Q\sinh(2\beta) + (\mu + p + \Pi)\sinh^2\beta\,,$$
$$\bar{p} = p - \frac{1}{3}Q\sinh(2\beta) + \frac{1}{3}(\mu + p + \Pi)\sinh^2\beta\,,$$
$$\bar{Q} = Q\cosh(2\beta) - \frac{1}{2}(\mu + p + \Pi)\sinh(2\beta)\,,$$
$$\bar{\Pi} = \Pi\left(1 + \frac{2}{3}\sinh^2\beta\right) - \frac{2}{3}Q\sinh(2\beta) + \frac{2}{3}(\mu + p)\sinh^2\beta\,,$$

$$\bar{\phi} = \phi\cosh\beta + \left(\frac{2}{3}\Theta + \Sigma\right)\sinh\beta$$
$$\bar{\mathcal{A}} = \mathcal{A}\cosh\beta + \left(\frac{1}{3}\Theta + \Sigma\right)\sinh\beta + (u^a\cosh\beta + e^a\sinh\beta)\nabla_a\beta$$
$$\bar{\theta} = \theta\cosh\beta + \left(\phi + \mathcal{A}\right)\sinh\beta + \nabla_u\cosh\beta + \nabla_e\sinh\beta\,,$$
$$\bar{\Sigma} = \Sigma\cosh\beta - \frac{1}{3}\left(\phi - 2\mathcal{A}\right)\sinh\beta + \frac{2}{3}\nabla_u\cosh\beta + \frac{2}{3}\nabla_e\sinh\beta\,,$$

# Change of frame

- In our case, $\beta$ can be proven to be first order. In fact

$$\beta \approx \frac{1}{\phi_0}\left(\Sigma - \frac{2}{3}\theta\right) = -\frac{\bar{Q}}{\mu_0 + p_0},$$

- Using this result we can derive the transformation of the perturbation variables, for example,

$$\mathsf{m} = \dot{\mu} = \nabla_{\bar{u}}\bar{\mu} + \frac{\widehat{\mu_0}}{\mu_0 + p_0}\bar{Q} = \bar{\mathsf{m}} + \frac{\widehat{\mu_0}}{\mu_0 + p_0}\bar{Q},$$

# Perturbed equation of state

- We assume that the *perturbed* source fluid is described in its rest frame by an equation of the state of the form

$$p = f(\mu)$$

- In the comoving frame this relation translate directly into

$$\mathsf{p} \approx f'(\mu_0)\,\mathsf{m}\,,$$

- To obtain the corresponding equations in the static frame we apply the transformations above

$$\bar{\mathsf{p}} \approx f'(\mu_0)\,\bar{\mathsf{m}} + \frac{1}{\mu_0 + p_0}\left(f'(\mu_0)\,\hat{\mu}_0 - \hat{p}_0\right)\bar{Q}\,,$$

# Boundary and Initial Conditions

- The junction conditions formulated in the comoving frame require at every instant

$$p\Big|_{\mathfrak{B}} = 0$$

- In the comoving frame this relation translate directly into

$$\mathsf{p}\Big|_{\mathfrak{B}} = 0$$

- The corresponding conditions in the static frame reads

$$\bar{\mathsf{p}} - \mathscr{A}_0 \bar{Q}\Big|_{\mathfrak{B}} = 0$$

- We also assume $\mathsf{m}$ and $\mathsf{p}$ finite in a neighborhood of the initial instant.

# Harmonic decomposition

- The perturbation equations we have derived are partial differential equations and, therefore not easy to solve
- A way to overcome this problem is to expand the solution in harmonics. This operation can be performed covariantly

$$\delta^2 Q^{(k)} = -\frac{k^2}{r^2} Q^{(k)}\,, \qquad \widehat{Q}^{(k)} = \dot{Q}^{(k)} = 0\,,$$

$$\frac{\hat{r}}{r} = \frac{1}{2}\phi\,, \qquad \frac{\dot{r}}{r} = \frac{1}{3}\theta - \frac{1}{2}\Sigma\,, \qquad \delta_\alpha r = 0\,,$$

$$\dot{T}^{(v)} = i v T^{(v)}\,, \qquad \widehat{T}^{(v)} = \delta_\alpha T^{(v)} = 0\,, \qquad \dot{v} = \delta_\alpha v = 0\,,$$

$$\widehat{v} = -\mathcal{A}_0 v\,, \qquad v\,(r) = \frac{\lambda}{\sqrt{(g_0)_{tt}}}$$

In our case, a generic scalar perturbation variable $\chi$ can be written as

$$\chi = \sum_{v^2=\{v_0^2, v_1^2, \dots\}} \Psi_\chi^{(v)}(r) Y_{00}\, e^{i v \tau} = \sum_{\lambda^2=\{\lambda_0^2, \lambda_1^2, \dots\}} \Psi_\chi^{(\lambda)}(r) Y_{00} e^{i\lambda t}$$

# Perturbations in the Comoving Frame

Using harmonic decomposition and adopting $r$ as radial parameter

$$\frac{d\Psi_{\mathsf{p}}^{(v)}}{dr}+\frac{4\mathcal{A}_0}{r\phi_0}\left(1+\frac{1}{3f'(\mu_0)}\right)\Psi_{\mathsf{p}}^{(v)}=-\frac{2(\mu_0+p_0)}{r\phi_0}\left(\Psi_{\mathsf{A}}^{(v)}+\mathcal{A}_0\Psi_{\Sigma}^{(v)}\right),$$

$$\frac{d\Psi_{\mathsf{A}}^{(v)}}{dr}+\left(\frac{6\mathcal{A}_0}{r\phi_0}-\frac{1}{r}\right)\Psi_{\mathsf{A}}^{(v)}=\frac{2\mathcal{E}_0}{r\phi_0(\mu_0+p_0)f'(\mu_0)}\Psi_{\mathsf{p}}^{(v)}-\frac{3}{r\phi_0}\left(v^2+\mathcal{A}_0^2+\frac{1}{3}\mu_0-2\mathcal{E}_0\right)\Psi_{\Sigma}^{(v)},$$

$$\frac{d\Psi_{\Sigma}^{(v)}}{dr}+\left(\frac{3}{r}-\frac{4\mathcal{A}_0}{3r\phi_0 f'(\mu_0)}\right)\Psi_{\Sigma}^{(v)}=\frac{2}{3(\mu_0+p_0)f'(\mu_0)}\left[\left(\frac{f''(\mu_0)}{f'(\mu_0)}+\frac{1}{\mu_0+p_0}\right)\frac{d\mu_0}{dr}+\frac{1}{r\phi_0}\left(\frac{4}{3f'(\mu_0)}+2\right)\mathcal{A}_0\right]\Psi_{\mathsf{p}}^{(v)}+\frac{4}{3r\phi_0 f'(\mu_0)}\Psi_{\mathsf{A}}^{(v)},$$

$$\left(v^2+\mathcal{A}_0\phi_0+\mathcal{A}_0^2-p_0\right)\left(\frac{2}{3}\Psi_{\theta}^{(v)}-\Psi_{\Sigma}^{(v)}\right)=\Psi_{\mathsf{p}}^{(v)}-\phi_0\Psi_{\mathsf{A}}^{(v)},\qquad \Psi_{\mathsf{F}}^{(v)}=\left(\frac{1}{2}\phi_0-\mathcal{A}_0\right)\left(\frac{2\Psi_{\mathsf{p}}^{(v)}}{3f'(\mu_0)(\mu_0+p_0)}+\Psi_{\Sigma}^{(v)}\right),$$

$$\Psi_{\mathsf{E}}^{(v)}=\mathcal{E}_0\left(\frac{3}{2}\Psi_{\Sigma}^{(v)}+\frac{\Psi_{\mathsf{p}}^{(v)}}{f'(\mu_0)(\mu_0+p_0)}\right)-\frac{1}{2}(\mu_0+p_0)\Psi_{\Sigma}^{(v)},\qquad \Psi_{\mathsf{m}}^{(v)}=-(\mu_0+p_0)\Psi_{\theta}^{(v)},\qquad \Psi_{\mathsf{p}}^{(v)}=f'(\mu_0)\Psi_{\mathsf{m}}^{(v)}$$

This system is associated with an eigenvalue problem similar the the Strum Liuville one…

# Analytic Exact Solutions

The perturbation system can be recast as:

$$\frac{d\mathbb{W}}{dr} = \left(r^{-1}\mathbb{R} + \Theta\right)\mathbb{W}\,, \qquad \mathbb{W} = \begin{bmatrix} \Psi_{\mathsf{p}}^{(\lambda)} \\ \Psi_{\mathsf{A}}^{(\lambda)} \\ \Psi_{\Sigma}^{(\lambda)} \end{bmatrix}, \quad \mathbb{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix},$$

$$\Theta = \frac{2}{r\phi_0}\begin{bmatrix} -2\mathcal{A}_0\left(1+\frac{1}{3f'(\mu_0)}\right) & -\left(\mu_0+p_0\right) & -\left(\mu_0+p_0\right)\mathcal{A}_0 \\ \frac{\mathcal{E}_0}{f'(\mu_0)(\mu_0+p_0)} & -3\mathcal{A}_0 & -\frac{3}{2}\left(v^2+\mathcal{A}_0^2+\frac{1}{3}\mu_0-2\mathcal{E}_0\right) \\ \frac{3f''(\mu_0)r\phi_0\partial_r\mu_0+4\mathcal{A}_0}{9(\mu_0+p_0)[f'(\mu_0)]^2}+\frac{r\phi_0\partial_r\mu_0+2\mathcal{A}_0(\mu_0+p_0)}{3(\mu_0+p_0)^2 f'(\mu_0)} & \frac{2}{3f'(\mu_0)} & \frac{2\mathcal{A}_0}{3f'(\mu_0)} \end{bmatrix}.$$

Suppose that:

1. the equilibrium fluid verifies the weak energy condition;
2. the background density and pressure are real analytic;
3. the square of the speed of sound $f'$ is positive and real analytic

# Analytic Exact Solutions

Using the background equations it is possible to write

$$\Theta\left(r\right) = \sum_{n=0}^{+\infty} \Theta_n r^n \,.$$

And the solution of perturbation system can be written as

$$\begin{bmatrix} \Psi_{\mathsf{p}}^{(\lambda)} \\ \Psi_{\mathsf{A}}^{(\lambda)} \\ \Psi_{\Sigma}^{(\lambda)} \end{bmatrix} = \begin{bmatrix} -1 & \frac{12}{r}\left[\left(\Theta_0\right)_{12}\left(\Theta_0\right)_{23} - 3\left(\Theta_1\right)_{13}\right] & 0 \\ 0 & 12\left(\Theta_0\right)_{23}\left[\left(\Theta_1\right)_{22} - \left(\Theta_1\right)_{33}\right] - 18\left(\Theta_2\right)_{23} + 4\left[\left(\Theta_0\right)_{23}\right]^2\left(\Theta_0\right)_{32} - \frac{12}{r^2}\left(\Theta_0\right)_{23} & r \\ 0 & \frac{36}{r^3} & 0 \end{bmatrix} \mathbb{P}_{\mathrm{W}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\mathbb{P}_{\mathrm{W}}\left(r\right) = \sum_{n=0}^{+\infty} \mathbb{P}_n r^n \,,$$

$$\mathbb{P}_0 = \mathbb{I}_3 \,,$$

$$\mathbb{P}_k = \frac{1}{k}\sum_{j=0}^{k-1} \mathbb{A}_{k-1-j}\mathbb{P}_j \,, \quad \text{for } k \geq 1 \,,$$

$$\mathbb{A}\left(r\right) = \sum_{n=0}^{+\infty} \mathbb{A}_n r^n$$

# Analytic Exact Solutions

With coefficients…

$$A_{11} = \Theta_{11} - \frac{1}{3} r^2 \Theta_{31} \left[ (\Theta_0)_{12} (\Theta_0)_{23} - 3 (\Theta_1)_{13} \right] ,$$

$$\begin{aligned} A_{12} = &- \frac{36\Theta_{13}}{r^3} + \frac{12}{r^2} \left[ \Theta_{12} (\Theta_0)_{23} - (\Theta_0)_{12} (\Theta_0)_{23} + 3 (\Theta_1)_{13} \right] + \frac{12}{r} (\Theta_{33} - \Theta_{11}) \left[ (\Theta_0)_{12} (\Theta_0)_{23} - 3 (\Theta_1)_{13} \right] \\ &+ \frac{2}{3} r^2 \Theta_{32} \left[ (\Theta_0)_{12} (\Theta_0)_{23} - 3 (\Theta_1)_{13} \right] \left\{ 6 (\Theta_0)_{23} \left[ (\Theta_1)_{22} - (\Theta_1)_{33} \right] - 9 (\Theta_2)_{23} + 2 (\Theta_0)_{32} \left[ (\Theta_0)_{23} \right]^2 \right\} \\ &- 4\Theta_{32} (\Theta_0)_{23} \left[ (\Theta_0)_{12} (\Theta_0)_{23} - 3 (\Theta_1)_{13} \right] - 12\Theta_{12} (\Theta_0)_{23} \left[ (\Theta_1)_{22} - (\Theta_1)_{33} \right] + 18\Theta_{12} (\Theta_2)_{23} \\ &- 4\Theta_{12} (\Theta_0)_{32} \left[ (\Theta_0)_{23} \right]^2 + 4r\Theta_{31} \left[ (\Theta_0)_{12} (\Theta_0)_{23} - 3 (\Theta_1)_{13} \right]^2 , \end{aligned}$$

$$A_{13} = \frac{1}{3} r^3 \Theta_{32} \left[ (\Theta_0)_{12} (\Theta_0)_{23} - 3 (\Theta_1)_{13} \right] - r\Theta_{12} ,$$

$$A_{21} = -\frac{1}{36} r^3 \Theta_{31} ,$$

$$\begin{aligned} A_{22} = &\frac{1}{9} r^3 \Theta_{32} \left\{ 3 (\Theta_0)_{23} \left[ (\Theta_1)_{22} - (\Theta_1)_{33} \right] - \frac{9}{2} (\Theta_2)_{23} + \left[ (\Theta_0)_{23} \right]^2 (\Theta_0)_{32} \right\} \\ &+ \frac{1}{3} r^2 \Theta_{31} \left[ (\Theta_0)_{12} (\Theta_0)_{23} - 3 (\Theta_1)_{13} \right] - \frac{r}{3} (\Theta_0)_{23} \Theta_{32} + \Theta_{33} , \end{aligned}$$

$$A_{23} = \frac{1}{36} r^4 \Theta_{32} ,$$

$$A_{31} = \frac{1}{9} \Theta_{31} \left\{ 3 (\Theta_0)_{23} \left[ r^2 (\Theta_1)_{22} - r^2 (\Theta_1)_{33} - 1 \right] - \frac{9}{2} r^2 (\Theta_2)_{23} + r^2 (\Theta_0)_{32} \left[ (\Theta_0)_{23} \right]^2 \right\} - \frac{\Theta_{21}}{r} ,$$

$$\begin{aligned} A_{32} = &- \frac{2r}{3} \Theta_{31} \left[ (\Theta_0)_{12} (\Theta_0)_{23} - 3 (\Theta_1)_{13} \right] \left\{ 6 (\Theta_0)_{23} \left[ (\Theta_1)_{22} - (\Theta_1)_{33} \right] - 9 (\Theta_2)_{23} + 2 \left[ (\Theta_0)_{23} \right]^2 (\Theta_0)_{32} \right\} \\ &- r^2 \Theta_{32} (\Theta_0)_{23} \left[ (\Theta_1)_{22} - (\Theta_1)_{33} \right] \left\{ 4 (\Theta_0)_{23} \left[ (\Theta_1)_{22} - (\Theta_1)_{33} \right] - 6 (\Theta_2)_{23} + \frac{4}{3} \left[ (\Theta_0)_{23} \right]^2 (\Theta_0)_{32} \right\} \\ &- r^2 \Theta_{32} \left[ (\Theta_0)_{23} \right]^2 (\Theta_0)_{32} \left\{ \frac{4}{3} (\Theta_0)_{23} \left[ (\Theta_1)_{22} - (\Theta_1)_{33} \right] - 2 (\Theta_2)_{23} + \frac{4}{9} \left[ (\Theta_0)_{23} \right]^2 (\Theta_0)_{32} \right\} \\ &+ \frac{36}{r^4} \left[ \Theta_{23} - (\Theta_0)_{23} \right] - \frac{12}{r^3} (\Theta_{22} - \Theta_{33}) (\Theta_0)_{23} + \frac{12}{r^2} \Theta_{21} \left[ (\Theta_0)_{12} (\Theta_0)_{23} - 3 (\Theta_1)_{13} \right] \\ &+ \frac{2}{r} (\Theta_{22} - \Theta_{33}) \left\{ 6 (\Theta_0)_{23} \left[ (\Theta_1)_{22} - (\Theta_1)_{33} \right] - 9 (\Theta_2)_{23} + 2 \left[ (\Theta_0)_{23} \right]^2 (\Theta_0)_{32} \right\} \\ &+ \frac{4}{3} (\Theta_0)_{23} \Theta_{32} \left\{ 6 (\Theta_0)_{23} \left[ (\Theta_1)_{22} - (\Theta_1)_{33} \right] - 9 (\Theta_2)_{23} + 2 \left[ (\Theta_0)_{23} \right]^2 (\Theta_0)_{32} \right\} \\ &+ r^2 \Theta_{32} (\Theta_2)_{23} \left\{ 6 (\Theta_0)_{23} \left[ (\Theta_1)_{22} - (\Theta_1)_{33} \right] - 9 (\Theta_2)_{23} + 2 \left[ (\Theta_0)_{23} \right]^2 (\Theta_0)_{32} \right\} \\ &+ \frac{2}{r^2} \left\{ 6 (\Theta_0)_{23} \left[ (\Theta_1)_{22} - (\Theta_1)_{33} \right] - 9 (\Theta_2)_{23} - 2 \left[ (\Theta_0)_{23} \right]^2 \left[ \Theta_{32} - (\Theta_0)_{32} \right] \right\} \\ &+ \frac{4}{r} \Theta_{31} (\Theta_0)_{23} \left[ (\Theta_0)_{12} (\Theta_0)_{23} - 3 (\Theta_1)_{13} \right] , \end{aligned}$$

$$A_{33} = \frac{r}{9} \Theta_{32} \left\{ 3 (\Theta_0)_{23} \left( 1 - r^2 \left[ (\Theta_1)_{22} - (\Theta_1)_{33} \right] \right) + \frac{9}{2} r^2 (\Theta_2)_{23} - \left[ (\Theta_0)_{23} \right]^2 (\Theta_0)_{32} r^2 \right\} + \Theta_{22} .$$

# Analytic Exact Solutions

Naturally we have to determine the radius of convergence of the series for each background, but there is no reason to think that it will not be as large as the radius of the relativistic star.

At lowest order we have

$$\begin{bmatrix} \Psi_{\mathsf{p}}^{(\lambda)} \\ \Psi_{\mathsf{A}}^{(\lambda)} \\ \Psi_{\Sigma}^{(\lambda)} \end{bmatrix} = \begin{bmatrix} -c_1 + \mathcal{O}\left(r^2\right) \\ c_3 r + \mathcal{O}\left(r^3\right) \\ \mathcal{O}\left(r^2\right) \end{bmatrix} .$$

where

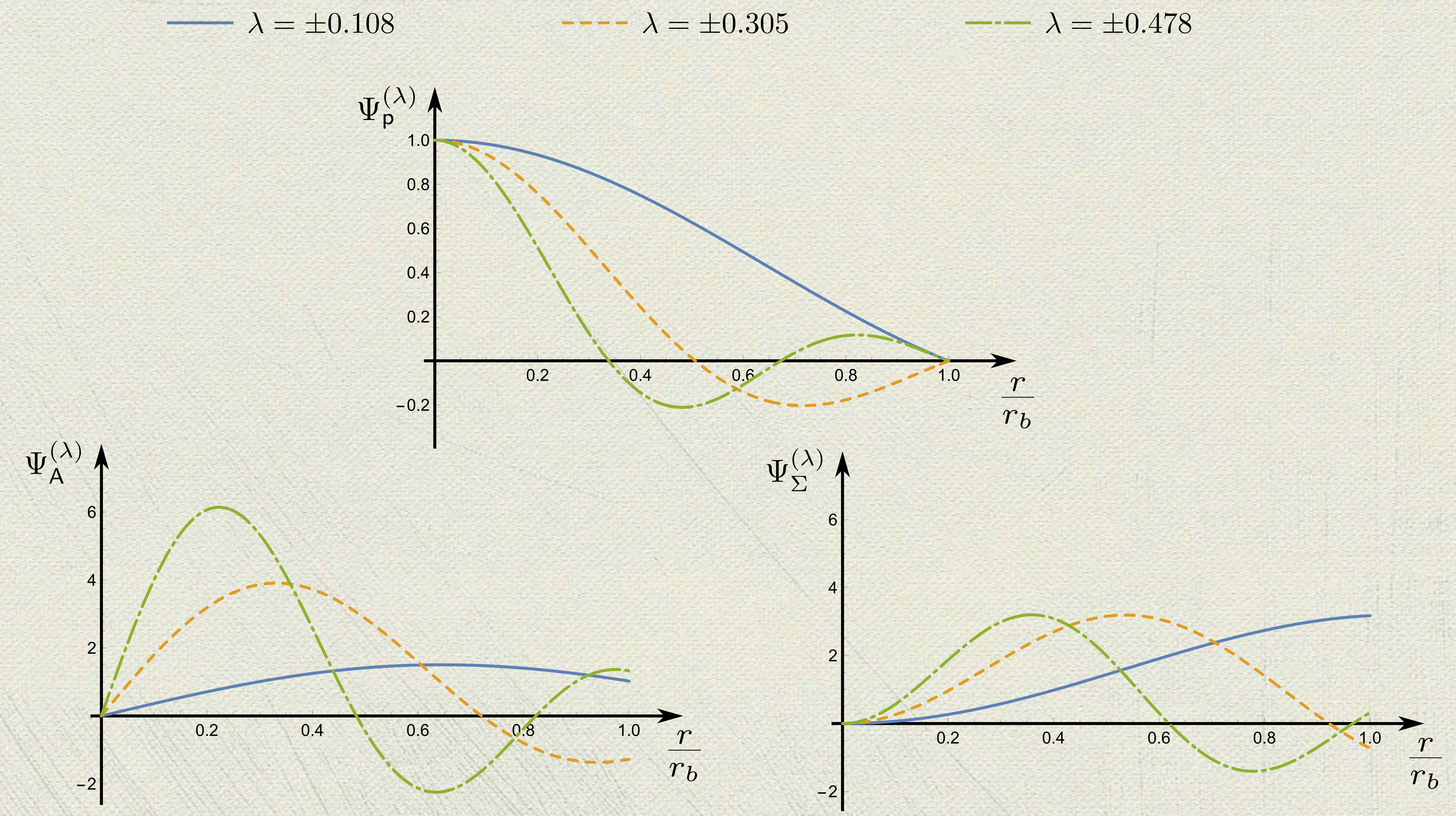
$$c_3 \overset{r=0}{\equiv} -\frac{c_1}{3\left(\mu_0 + p_0\right) f'\left(\mu_0\right)} \left[\frac{\lambda^2}{\left(g_0\right)_{tt}} + \frac{1}{3}\mu_0 + \frac{3}{2}\left(\mu_0 + p_0\right) f'\left(\mu_0\right)\right]$$

which relates the perturbation at the boundary to the eigenfrequency $\lambda$

# Examples

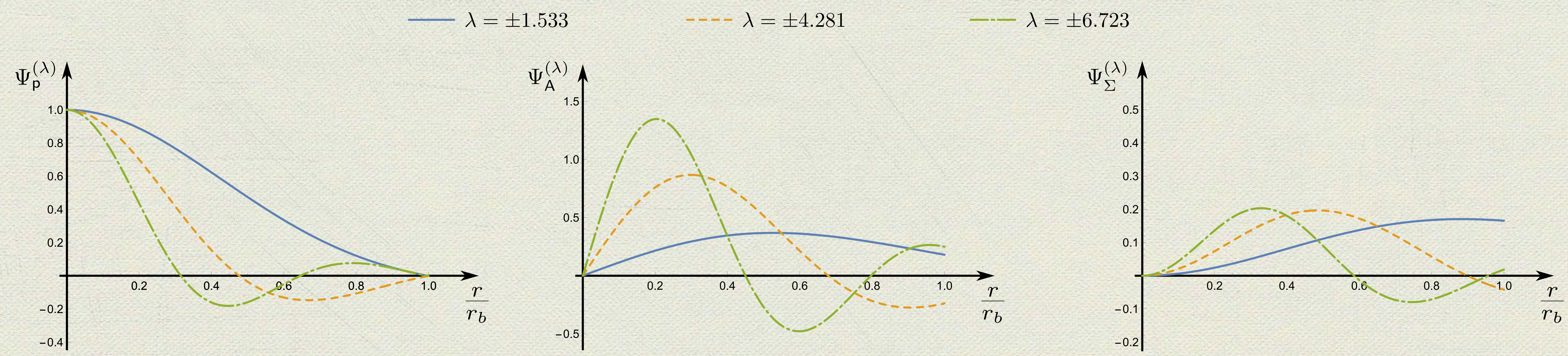
| Spacetime | Non-trivial metric components |
|---|---|
| Interior Schwarzschild | $(g_0)_{tt} = \left(3\sqrt{1-\frac{2M}{r_b}} - \sqrt{1-\frac{2Mr^2}{r_b^3}}\right)^2$ <br> $(g_0)_{rr} = \left(1-\frac{2Mr^2}{r_b^3}\right)^{-1}$ |
| Tolman IV | $(g_0)_{tt} = B^2\left(\frac{r^2}{A^2}+1\right)$ <br> $(g_0)_{rr} = \frac{\frac{2r^2}{A^2}+1}{\left(1+\frac{r^2}{A^2}\right)\left(1-\frac{r^2}{R^2}\right)}$ |
| Kuch2 III | $(g_0)_{tt} = Be^{\frac{Ar^2}{2}}$ <br> $(g_0)_{rr} = \left(r^2 e^{-\frac{1}{2}Ar^2}\left[C - \frac{A}{2e}\mathrm{Ei}\left(\frac{Ar^2}{2}+1\right)\right]+1\right)^{-1}$ |
| Heint IIa | $(g_0)_{tt} = A^2\left(ar^2+1\right)^3$ <br> $(g_0)_{rr} = \left(1 - \frac{3ar^2\left[c\left(4ar^2+1\right)^{-\frac{1}{2}}+1\right]}{2\left(ar^2+1\right)}\right)^{-1}$ |

| Spacetime | Parameters | $\lvert\lambda_0\rvert$ | $\lvert\lambda_1\rvert$ | $\lvert\lambda_2\rvert$ |
|---|---|---|---|---|
| Interior Schwarzschild | $(M, r_b, c_s^2) = (0.1, 1, 0.1)$ | 0.108 | 0.305 | 0.478 |
| Tolman IV | $(A, B, R) = (1, 1, 1.5)$ | 1.533 | 4.281 | 6.723 |
| Kuch2 III | $(A, B, C) = (5, 1, -3)$ | 20.214 | 41.085 | 61.808 |
| Heint IIa | $(a, A, C) = (1, 1, 1.5)$ | 4.004 | 10.262 | 15.939 |

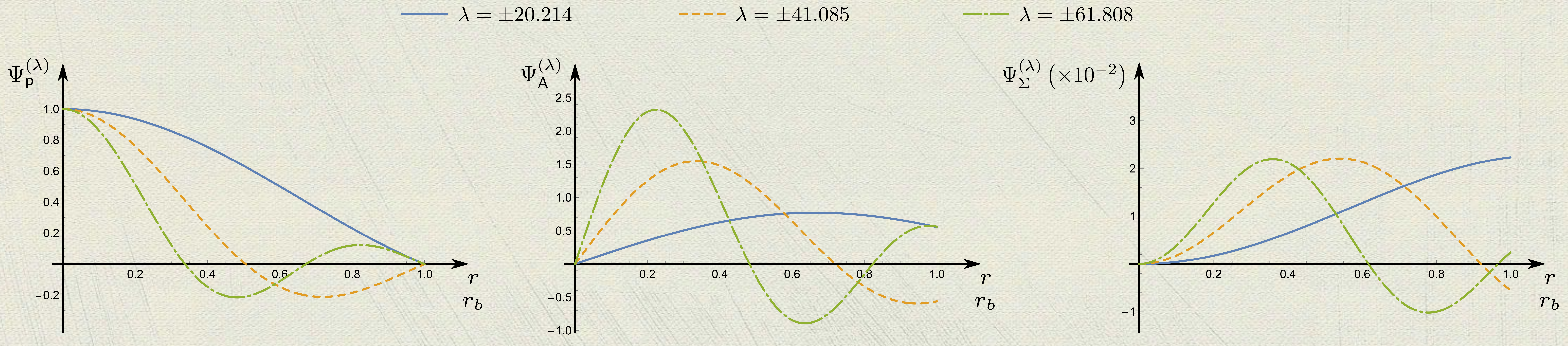


# Examples

Tolman IV

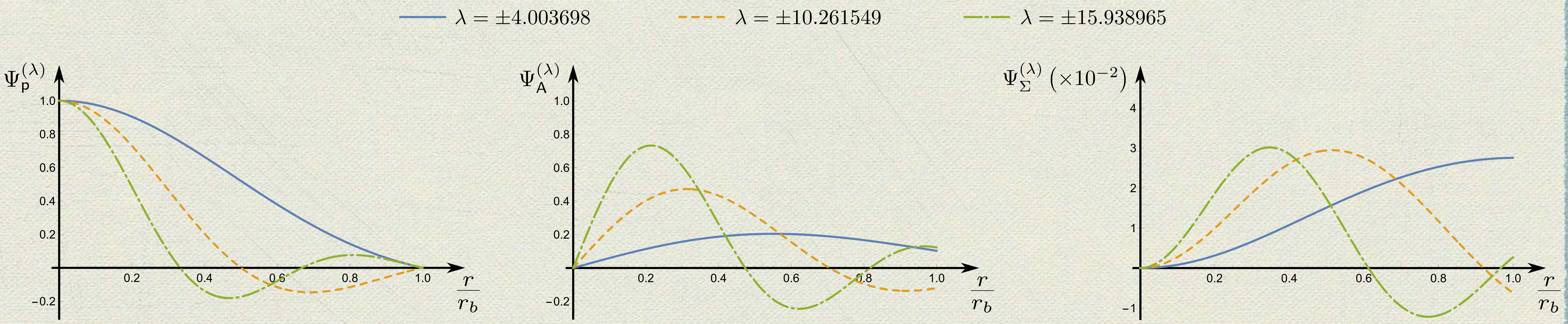



Kuchowicz2III



# Examples

Heintzmann IIa




All solutions represented here are stable for the parameters chosen.
(but instability could be determined without problems)

# Perturbations in the static frame

Using harmonic decomposition and adopting $r$ as radial parameter

$$\frac{d\Psi_{\mathsf{p}}^{(\upsilon)}}{dr} = \frac{2}{r\phi_0}\left[\frac{\mu_0+p_0}{\phi_0}\left(\frac{1}{2}\phi_0+2\mathcal{A}_0\right)+\frac{\mathcal{A}_0^2}{f'(\mu_0)}+\frac{r\mathcal{A}_0\phi_0}{2(\mu_0+p_0)}\frac{d\mu_0}{dr}+\upsilon^2\right]\Psi_Q^{(\upsilon)} - \frac{2}{r\phi_0}\left[\frac{\mu_0+p_0}{\phi_0}+\left(2+\frac{1}{f'(\mu_0)}\right)\mathcal{A}_0\right]\Psi_{\mathsf{p}}^{(\upsilon)}$$

$$\frac{d\Psi_Q^{(\upsilon)}}{dr} = \frac{2}{r\phi_0}\left[\frac{\mathcal{A}_0}{f'(\mu_0)}+\frac{r\phi_0}{2(\mu_0+p_0)}\frac{d\mu_0}{dr}+\frac{\mu_0+p_0}{\phi_0}-\phi_0-2\mathcal{A}_0\right]\Psi_Q^{(\upsilon)} - \frac{2}{r\phi_0 f'(\mu_0)}\Psi_{\mathsf{p}}^{(\upsilon)}$$

$$\Psi_{\mathsf{m}}^{(\upsilon)} = \frac{1}{f'(\mu_0)}\Psi_{\mathsf{p}}^{(\upsilon)} - \left(\frac{\mathcal{A}_0}{f'(\mu_0)}+\frac{r\phi_0}{2(\mu_0+p_0)}\frac{d\mu_0}{dr}\right)\Psi_Q^{(\upsilon)}\,, \qquad \Psi_\theta^{(\upsilon)} = -\frac{1}{\phi_0}\Psi_Q^{(\upsilon)}\,,$$

$$\Psi_{\mathsf{A}}^{(\upsilon)} = \frac{1}{\phi_0}\left[\Psi_{\mathsf{p}}^{(\upsilon)} - \left(\frac{1}{2}\phi_0+\mathcal{A}_0\right)\Psi_Q^{(\upsilon)}\right]\,, \qquad \Psi_\Sigma^{(\upsilon)} = \frac{2}{3}\Psi_\theta^{(\upsilon)}\,,$$

$$\Psi_{\mathsf{E}}^{(\upsilon)} = \frac{1}{2}\phi_0\Psi_Q^{(\upsilon)} + \frac{1}{3f'(\mu_0)}\Psi_{\mathsf{p}}^{(\upsilon)}\,, \qquad \Psi_{\mathsf{F}}^{(\upsilon)} = \Psi_Q^{(\upsilon)}\,,$$

Which constitutes a limit-point-non-oscillating (LPNO) endpoint Sturm-Liuville problem

# Analytic Exact Solutions

As before the perturbation system can be recast as:

$$\frac{d\mathbb{W}}{dr} = \left(r^{-1}\mathbb{R} + \Theta\right)\mathbb{W}\,, \qquad \mathbb{W} = \begin{bmatrix} \Psi_{\mathsf{p}} \\ \Psi_Q \end{bmatrix}, \qquad \mathbb{R} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix},$$

$$\Theta = -\frac{2}{r\phi_0}\begin{bmatrix} \frac{\mu_0+p_0}{\phi_0} + 2\mathcal{A}_0 + \frac{\mathcal{A}_0}{f'(\mu_0)} & -\frac{\mu_0+p_0}{\phi_0}\left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right) - \frac{\mathcal{A}_0^2}{f'(\mu_0)} - \frac{r\mathcal{A}_0\phi_0}{2(\mu_0+p_0)}\frac{d\mu_0}{dr} - \upsilon^2 \\ \frac{1}{f'(\mu_0)} & 2\mathcal{A}_0 - \frac{\mathcal{A}_0}{f'(\mu_0)} - \frac{r\phi_0}{2(\mu_0+p_0)}\frac{d\mu_0}{dr} - \frac{\mu_0+p_0}{\phi_0} \end{bmatrix}.$$

Suppose that:

1. the equilibrium fluid verifies the weak energy condition;
2. the background density and pressure are real analytic;
3. the square of the speed of sound $f'$ is positive and real analytic

# Analytic Exact Solutions

Using the background equations it is possible to write

$$\Theta(r) = \sum_{n=0}^{+\infty} \Theta_n r^n \,.$$

And the solutions of the perturbation system can be written as

$$\begin{bmatrix} \Psi_{\mathsf{p}} \\ \Psi_Q \end{bmatrix} = \begin{bmatrix} -\frac{1}{r}(\Theta_0)_{12} & 1 \\ \frac{1}{r^2} & 0 \end{bmatrix} \mathbb{P}_{\mathbb{W}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\mathbb{P}_{\mathbb{W}}(r) = \sum_{n=0}^{+\infty} \mathbb{P}_n r^n$$

$$\mathbb{P}_0 = \mathbb{I}_2 \,,$$

$$\mathbb{P}_k = \frac{1}{k}\sum_{j=0}^{k-1} \mathbb{A}_{k-1-j}\mathbb{P}_j \,, \quad \text{for } k \geq 1$$

$$\mathbb{A}(r) = \sum_{n=0}^{+\infty} \mathbb{A}_n r^n$$

# Analytic Exact Solutions

With coefficients…

$$\mathbb{A} = \left[\begin{array}{cc} \Theta_{22} - r\left(\Theta_0\right)_{12}\Theta_{21} & r^2\Theta_{21} \\ \frac{\Theta_{12}-\left(\Theta_0\right)_{12}}{r^2} + \frac{\left(\Theta_0\right)_{12}\left(\Theta_{22}-\Theta_{11}\right)}{r} - \left(\Theta_0\right)_{12}^2\Theta_{21} & \Theta_{11} + r\left(\Theta_0\right)_{12}\Theta_{21} \end{array}\right],$$

The equations that  describe the perturbations are much simpler!

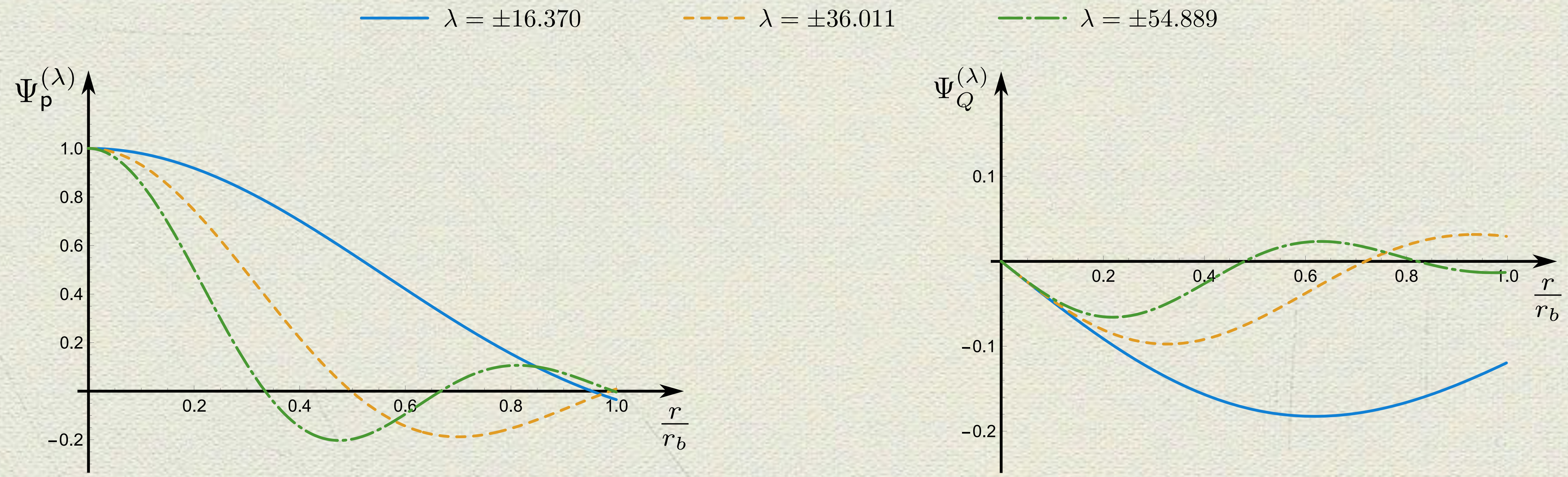
Also in this case can easily a lower bound for the eigenvalues

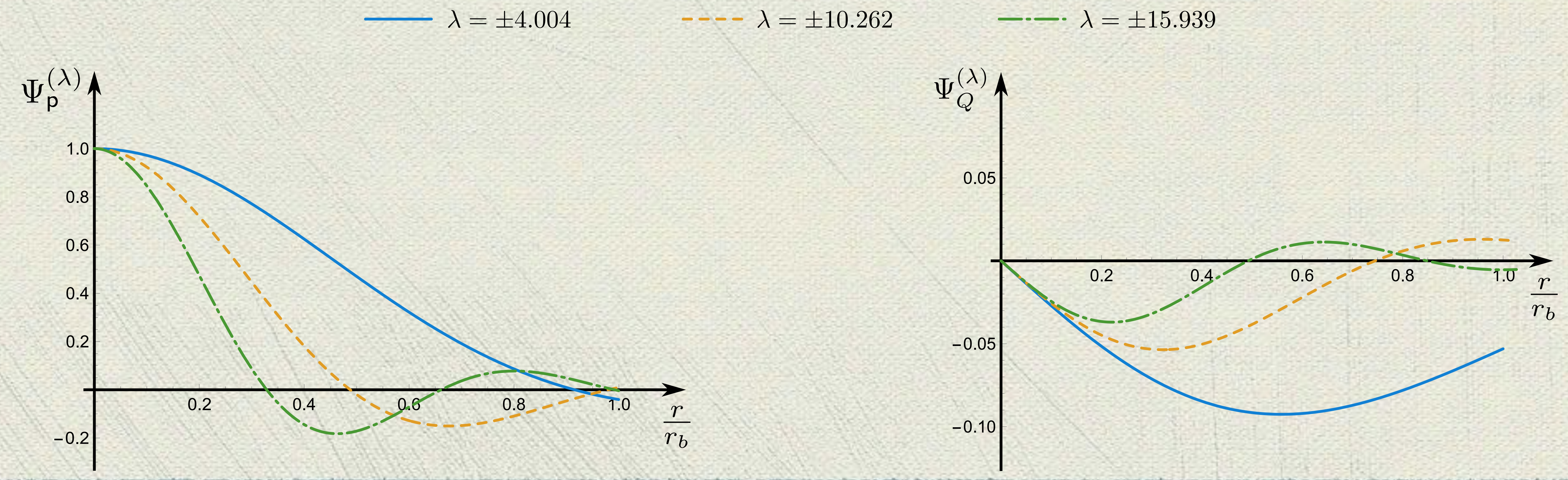
$$\lambda^2 \max_{r\in]0,r_b[} \left(g_0\right)_{tt} > - \max_{r\in]0,r_b[} \left[\frac{\mu_0 + p_0}{\phi_0}\left(\frac{1}{2}\phi_0 + 2\mathcal{A}_0\right) + \frac{\mathcal{A}_0^2}{f'\left(\mu_0\right)} + \frac{r\phi_0\mathcal{A}_0}{2\left(\mu_0 + p_0\right)}\frac{d\mu_0}{dr}\right]$$

# Examples

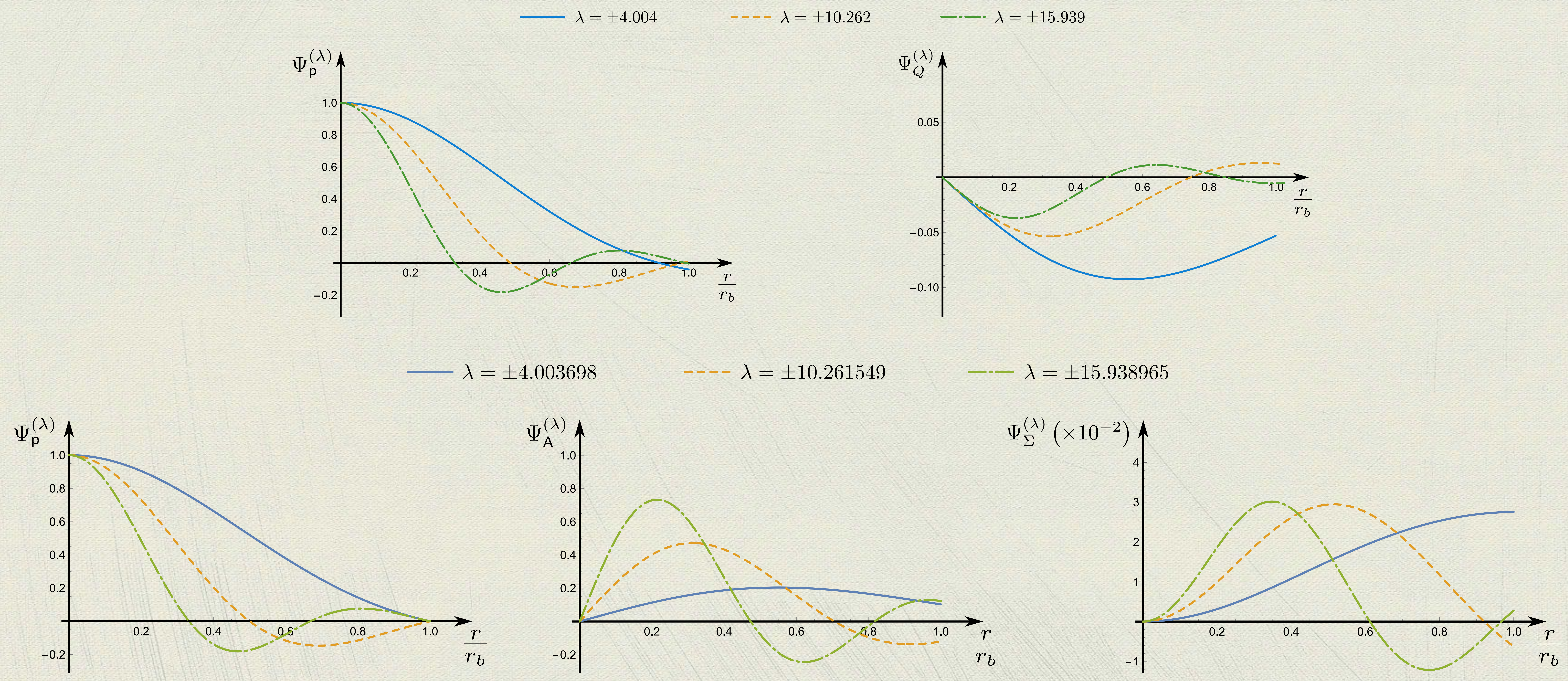
| Spacetime | Non-trivial metric components |
|---|---|
| Buch1 | $(g_0)_{tt} = A\left[\left(1+Cr^2\right)^{\frac{3}{2}} + B\left(5+2Cr^2\right)\sqrt{2-Cr^2}\right]^2$<br>$(g_0)_{rr} = \frac{2\left(1+Cr^2\right)}{2-Cr^2}$ |
| Heint IIa | $(g_0)_{tt} = A^2\left(ar^2+1\right)^3$<br>$(g_0)_{rr} = \left(1 - \frac{3ar^2\left[c\left(4ar^2+1\right)^{-\frac{1}{2}}+1\right]}{2\left(ar^2+1\right)}\right)^{-1}$ |
| P-V IIa | $(g_0)_{tt} = \left\{A\cos\left[\frac{1}{2}\text{arcsinh}\left(\frac{b^2r^2-c}{\sqrt{b^2-c^2}}\right)+d\right] + B\sin\left[\frac{1}{2}\text{arcsinh}\left(\frac{b^2r^2-c}{\sqrt{b^2-c^2}}\right)+d\right]\right\}^2$<br>$(g_0)_{rr} = \left(b^2r^4-2cr^2+1\right)^{-1}$ |
| Tolman VII | $(g_0)_{tt} = B^2\sin^2\left[\ln\left(\sqrt{\frac{\sqrt{1-\frac{r^2}{R^2}+\frac{4r^4}{A^4}}+\frac{2r^2}{A^2}-\frac{A^2}{4R^2}}{C}}\right)\right]$<br>$(g_0)_{rr} = \left(1-\frac{r^2}{R^2}+\frac{4r^4}{A^4}\right)^{-1}$ |

| Spacetime | Parameters | $\|\lambda_0\|$ | $\|\lambda_1\|$ | $\|\lambda_2\|$ |
|---|---|---|---|---|
| Buch1 | $(A,B,C)=(1,0.5,1)$ | 16.370 | 36.011 | 54.889 |
| Heint IIa | $(a,A,C)=(1,1,1.5)$ | 4.004 | 10.262 | 15.939 |
| P-V IIa | $(A,B,b,c,d)=(1,3,2,1,1)$ | 8.192 | 18.105 | 27.624 |
| Tolman VII | $(A,B,C,R)=(1,1,20,0.54)$ | 3.434 | 7.906 | 12.120 |



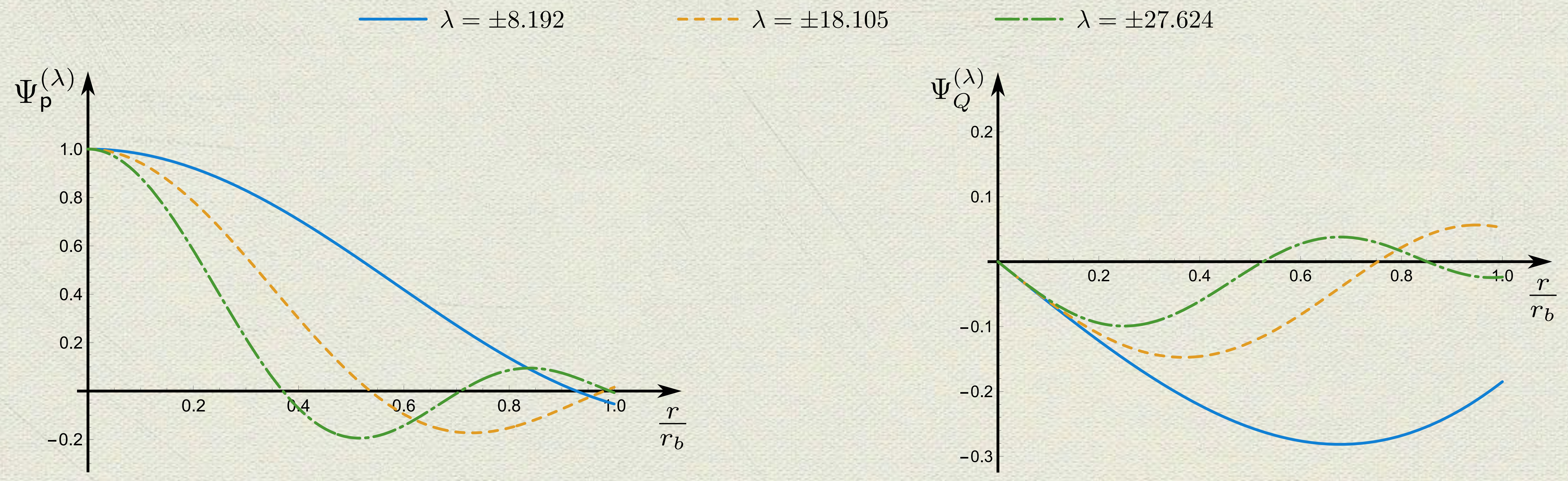



# Examples: Hentzmann IIa

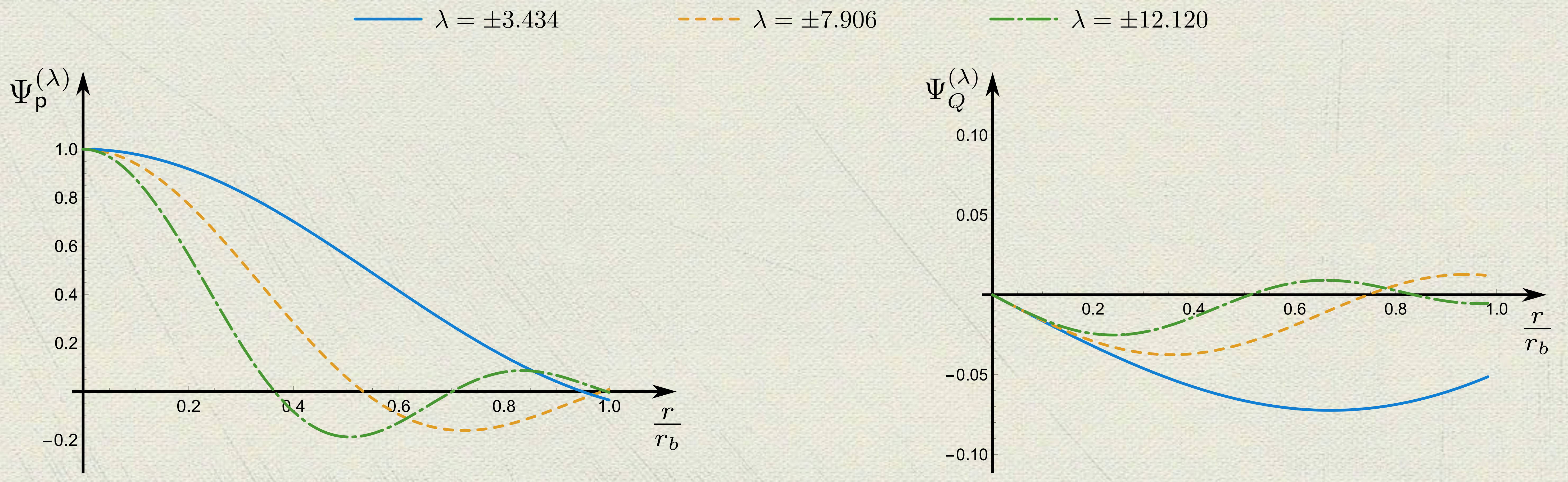


# Examples

Patwardhan-Vaidya IIa




Tolman VII



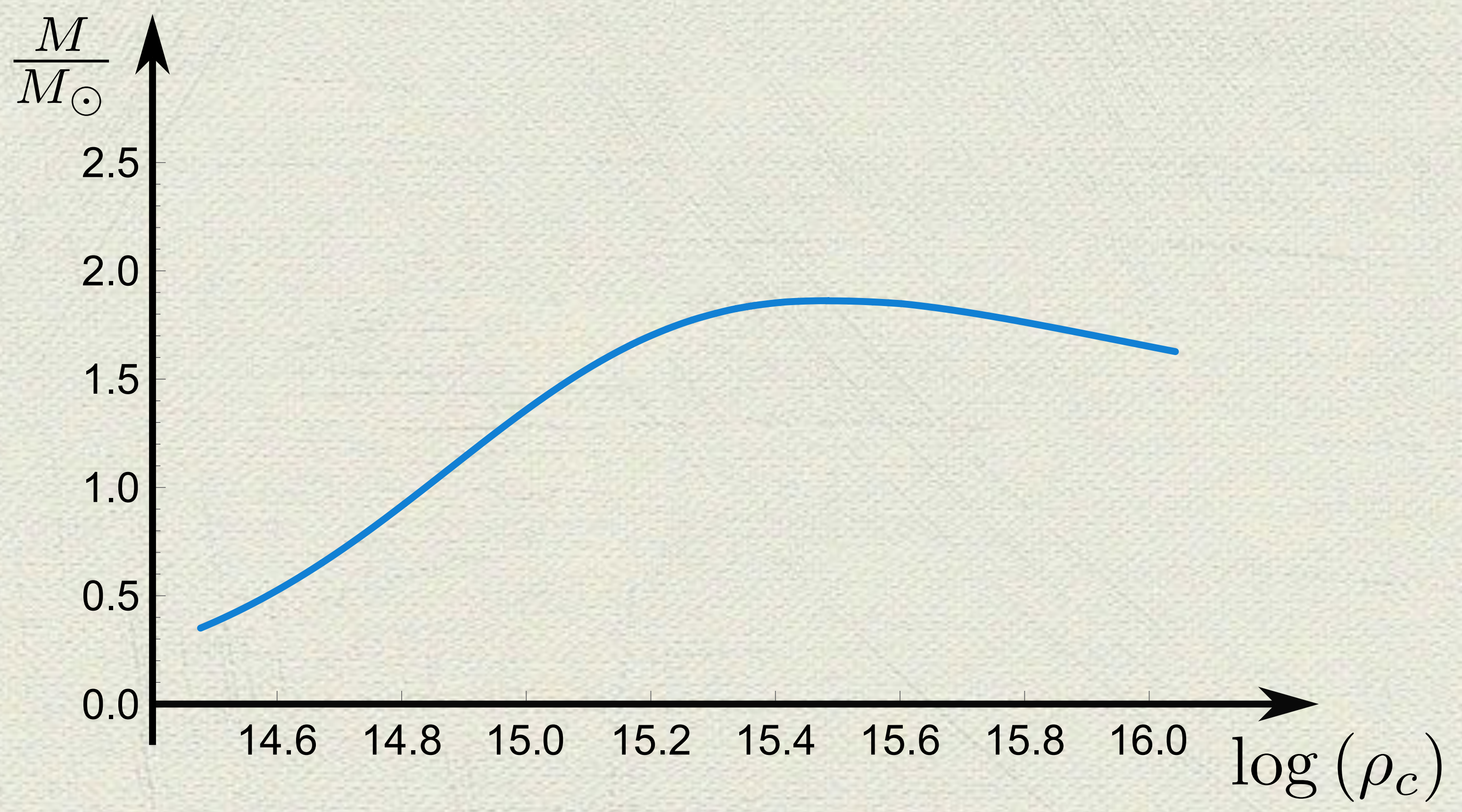

All solutions represented here are stable for the parameters chosen.

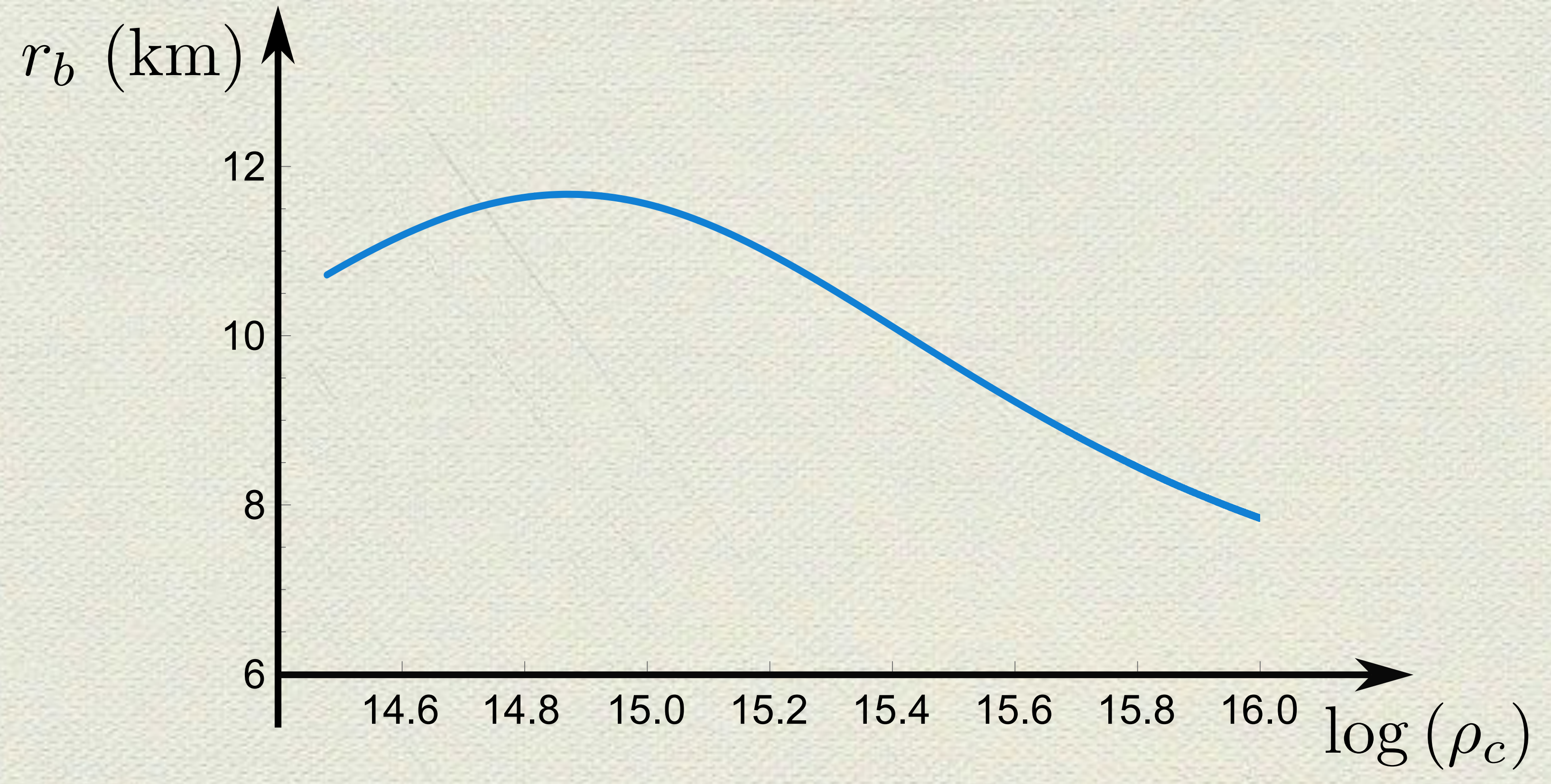
# Numerics: the Bethe-Johnson EoS

- The results above are analytical but require a particularly regular background.
- This is not the case for more realistic models of relativistic stars.
- The new perturbation equations, however, can be implemented numerically and they are less problematic than the Chandrasekhar equation.
- We can test this property by considering the Bethe-Johnson EoS (in SI units)

$$E = 236 n^{1.54} + m_n \,,$$
$$p = 363.44 n^{2.54} \,,$$
$$0.1 \lesssim n \lesssim 3 \text{fm}^{-3}$$
$$1.7 \times 10^{14} \lesssim \rho \lesssim 1.1 \times 10^{16} \text{ g/cm}^3 \,.$$

# Numerics: the Bethe-Johnson EoS




| $\rho_c$ $\left(10^{15}\text{g/cm}^3\right)$ | $r_b$ (km) | $\frac{M}{M_\odot}$ | $\frac{GM}{c^2 r_b}$ | $f_1$ (kHz) | $f_2$ (kHz) | $f_3$ (kHz) |
|---|---|---|---|---|---|---|
| 3.10 | 9.692 | 1.864 | 0.284 | 1.066* | 18.658 | 28.835 |
| 3.05 | 9.724 | 1.865 | 0.283 | 0.647 | 18.678 | 28.536 |
| 2.80 | 9.891 | 1.862 | 0.278 | 2.366 | 17.550 | 27.008 |
| 2.50 | 10.115 | 1.851 | 0.270 | 3.270 | 16.378 | 25.093 |
| 2.00 | 10.545 | 1.801 | 0.252 | 3.998 | 14.266 | 21.678 |
| 1.50 | 11.058 | 1.669 | 0.223 | 4.147 | 11.916 | 17.940 |
| 1.00 | 11.554 | 1.358 | 0.174 | 3.800 | 9.271 | 13.809 |

# Paying the Price in Thermodynamics…

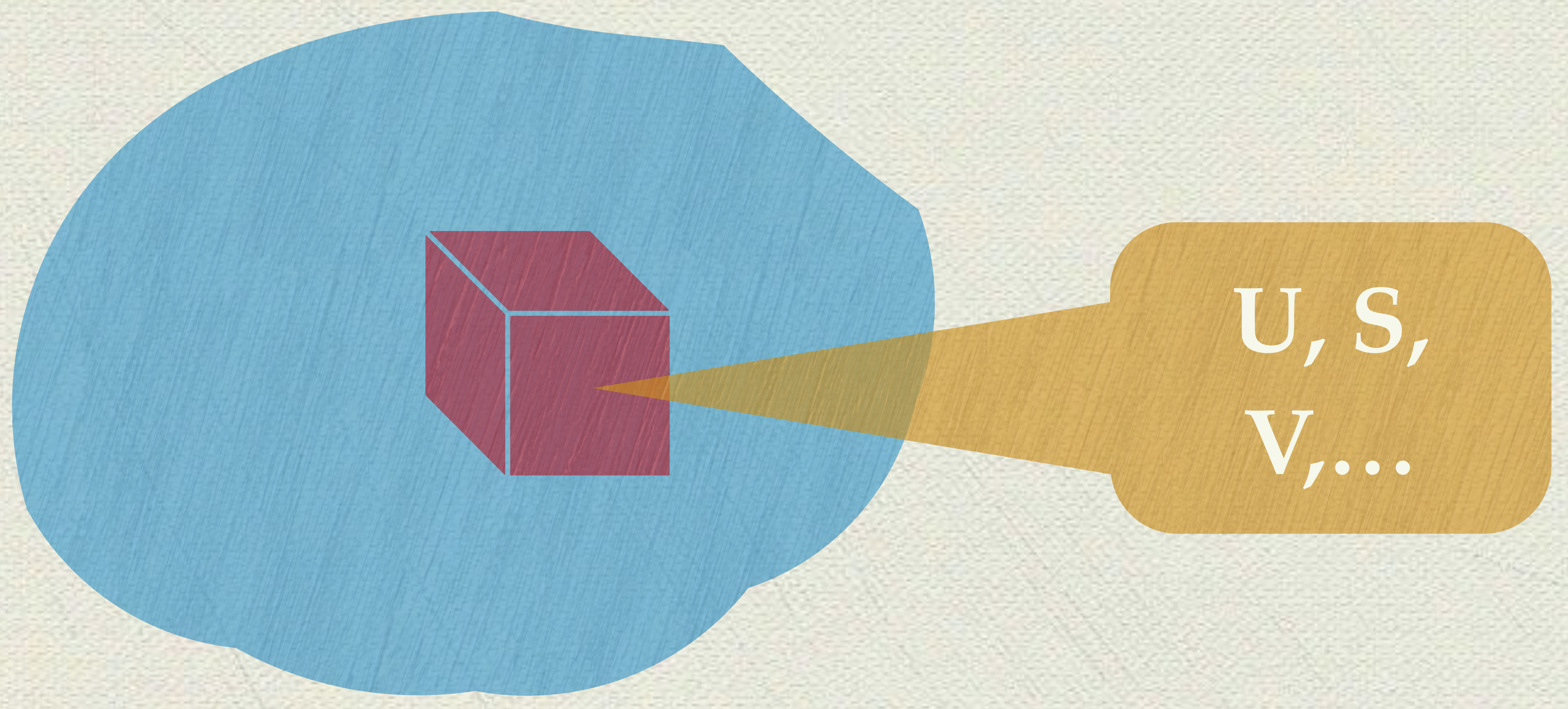
- The perturbation equations in the static frame are simpler than the comoving one, but the thermodynamics of the fluid becomes non trivial.
- In particular, the fluid acquires a matter flux given by

$$\bar{Q} = -(\mu + p)\,\beta\,,$$

- Fluxes of this type are normally associated with nonadiabatic processes. Can a change of frame change the character of a perturbative thermodynamical process?
- The answer is no (at first order), but it is still worth to see why…

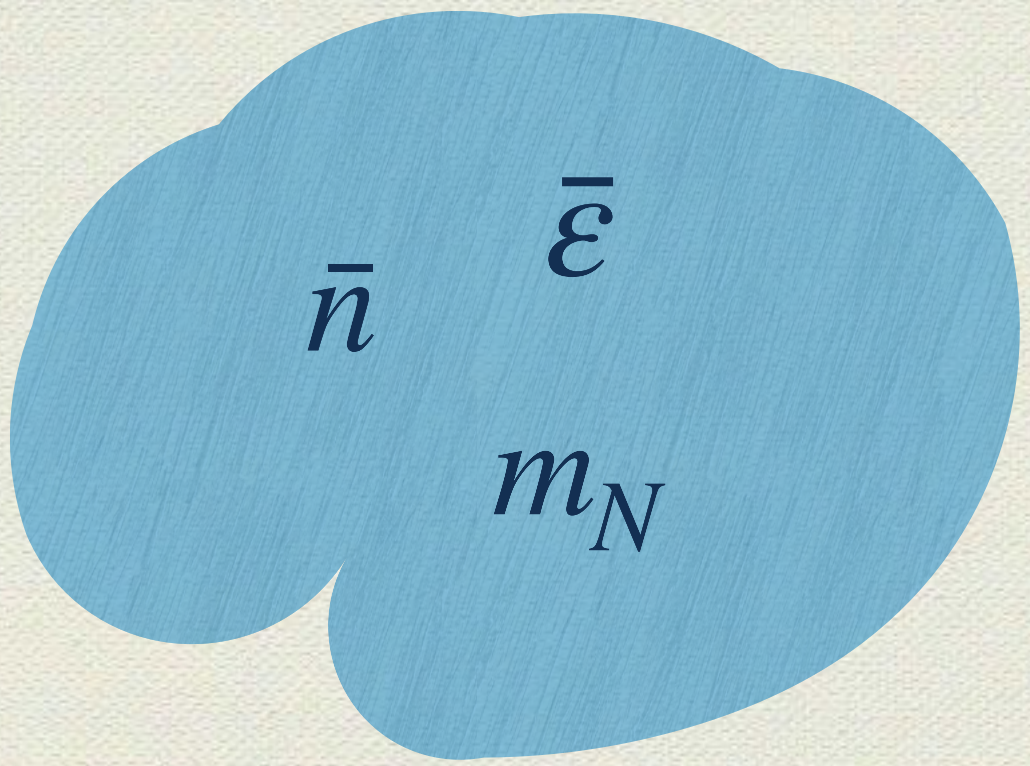
# Relativistic Thermodynamics

- The thermodynamics of a relativistic fluid in equilibrium is described by $T_{ab}$, $N^a$, $S^a$
- However, the perturbed spacetimes are not in equilibrium…
- As we are in a perturbative regime, we can employ the local thermodynamical ansatz




- One way to understand if adiabaticity is preserved by the change of frame is to derive the first law of thermodynamics as seen by the static observer

# Relativistic Thermodynamics




- Given a fluid, we define the rest mass energy, the specific volume, and the energy density

$$\bar{\mu}_N = m_N \bar{n} \qquad \bar{v} = 1/\bar{\mu}_N \qquad \bar{\mu} = \bar{\mu}_N (1 + \bar{\varepsilon})$$

- Using the general form of the first principle of thermodynamics,

$$d\bar{\varepsilon} = -\delta\overline{\mathsf{W}} + \delta\bar{\mathsf{Q}} = -\bar{p}\,d\bar{v} + \overline{T}\,d\bar{S}$$

- The change in energy density due to a quasi-equilibrium thermodynamical process can be written

$$d\bar{\mu} = -\bar{\mu}\frac{d\bar{v}}{\bar{v}} + \bar{\mu}_N d\bar{\varepsilon} = (\bar{\mu} + \bar{p})\frac{d\bar{n}}{\bar{n}} + \bar{\mu}_N \overline{T}\,d\bar{S}$$

# Relativistic Thermodynamics

The particle 4-current density vector in the comoving frame is given by

$$N^\alpha = n u^\alpha$$

Given an isotropic change of frame,

$$\bar{u}^a \approx u^a + e^a \beta \,,$$
$$\bar{e}^a \approx u^a \beta + e^a \,,$$

$$N^a = n u^a \to \bar{N}^a \approx = \bar{n}\bar{u}^a + \bar{n}^a$$
$$\bar{n}^a = n \beta \bar{e}^a$$

The conservation of particle number gives

$$\nabla_a \bar{N}^a = \bar{u}^\alpha \nabla_\alpha \bar{n} + \bar{n}\bar{\theta} + \nabla_\alpha \bar{n}^\alpha = 0$$

$$\frac{\dot{\bar{n}}}{\bar{n}} = -\bar{\theta} - \frac{1}{\bar{n}} \nabla_\alpha \bar{n}^\alpha \qquad \frac{1}{\bar{n}} \nabla_\alpha \bar{n}^\alpha = \frac{\phi_0 + 2\mathscr{A}_0}{\mu_0 + p_0} \bar{Q} + \frac{\widehat{\bar{Q}}}{\mu_0 + p_0}$$

# Heat Flow at First Order

Using the relations above we can write an equation for the time evolution of the energy density

$$\dot{\mu} = -\left(\mu_0 + p_0\right)\bar{\theta} - \left(\phi_0 + 2\mathcal{A}_0\right)\bar{Q} - \hat{\bar{Q}} + \bar{\mu}_N \overline{T}\dot{\bar{S}}$$

Which can be compared with the full 1+1+2 conservation law

$$\dot{\mu} = -\left(\mu_0 + p_0\right)\bar{\theta} - \left(\phi_0 + 2\mathcal{A}_0\right)\bar{Q} - \hat{\bar{Q}} + \text{higher-order terms}$$

Whic implies that

$$\int_c \delta\bar{\mathsf{Q}} = \int_{\tau_i}^{\tau_f} \overline{T}\dot{\bar{S}}\, d\tau = \text{higher-order terms}$$

Thus there is no exchange of energy as heat at linear level.

# Entropy and Temperature

In spite of this results, the static observer sees a convection term in her entropy flow. In fact

$$S^a = m_N S n u^a \to \bar{S}^a \approx m_N S n \bar{u}^a - s^a$$
$$s^a = m_N S \bar{e}^a \beta$$

Using the expression of $\beta$ derived above

$$\bar{s}^\alpha = \frac{m_N n_0 S_0}{\mu_0 + p_0} \bar{Q} \bar{e}^\alpha$$

As convection terms are usually written in terms of the temperature we obtain

$$\bar{s}^\alpha \propto \frac{\eta \bar{Q} \bar{e}^\alpha}{\overline{T}} \Rightarrow \overline{T} = \frac{\mu_0 + p_0}{m_N n_0 S_0}$$

# Conclusions

- Chandrasekhar pulsation equation can be generalized in a form that is covariant and gauge invariant
- The new "pulsation equations" can be written in any reference frame and do not require a trial function to determine stability.
- We have written the pulsation equations in two frames: comoving and static
- In the comoving frame thermodynamics is easier but the perturbation equations are more complicated
- In the static frame, the perturbation equations are simpler, but the thermodynamics is more complicated.
- Yet, the static frame equations are an interesting laboratory to work on relativistic thermodynamics out of equilibrium.
- In both frames, the perturbation equations can be solved both analytically (by series) and numerically (non-stiff system)

“No AIs were harmed during the preparation of this presentation”