Gravitational Waves Induced by Scalar-tensor Mixing

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P. Bari, N. Bartolo, G. Domènech, and S. Matarrese, "Gravitational waves induced by scalar-tensor mixing", Phys. Rev. D 109, 023509 (2024).

Introduction

- Investigating primordial gravitational waves is a cornerstone objective in contemporary cosmology.
- PTA collaboration's July 29, 2023 revelation of a GWs background prompts deeper inquiry $\rightarrow$ scalar-induced gravitational waves (SIGWs) background emerging as a compelling candidate.
- Non-linear interactions between scalar perturbations and GWs $\rightarrow$ SIGWs (Tomita 1967, Matarrese *et. al* 1998...for a review, see Domènech 2021)

Motivation

- SIGWs are the primary expectation for secondary gravitational wave signals, yet interactions between scalar-tensor and tensor-tensor perturbations should also be taken into consideration (e.g. J.-O. Gong 2019).
- Z. Chang *et al.* 2023 $\rightarrow$ with monochromatic primordial scalar and tensor spectra, the scalar-tensor induced GWs may dominate the high-k regime of the total induced GWs.

Can this effect be observed over SIGWs?

Set-up: Scalar Induced Gravitational Waves

- Perturbed flat FLRW space-time in radiation domination

$$ds^2 = -N^2 dt^2 + a^2 e^{-2\Psi} \left(e^{\gamma}\right)_{ij} dx^i dx^j \,,$$

where $N = e^{\Phi}$, Φ, $\Psi \rightarrow$ scalar perturbations, $\gamma_{ij} \rightarrow$ tensor perturbations.

- Taking only scalar-scalar terms as the source, the evolution equation becomes

$$\ddot{\gamma}^i_k + 3H\dot{\gamma}^i_k - \frac{\nabla^2 \gamma^i_k}{a^2} = \frac{4}{a^2}\Phi^{,i}\Phi_{,k} + 16\pi G(\overline{\rho} + \overline{P})v^{,i}v_{,k} \,.$$

Scalar Induced Gravitational Waves (SIGWs)

In conformal time and Fourier space

$$\gamma_{\boldsymbol{k}}'' + 2\mathcal{H}\gamma_{\boldsymbol{k}}' + k^2\gamma_{\boldsymbol{k}} = 4\int \frac{d^3k_1}{(2\pi)^3}\Phi_{\boldsymbol{k}}(0)\Phi_{\boldsymbol{k}-\boldsymbol{k_1}}(0)\epsilon_i^k(\hat{\boldsymbol{k}})k_1^i k_{1k}$$

$$\Big[T_\Phi(k_1\eta)T_\Phi(|\boldsymbol{k}-\boldsymbol{k_1}|\eta) + \frac{1}{2}\left(\mathcal{H}T_\Phi(k_1\eta) + T_\Phi'(k_1\eta)\right)$$

$$\left(\mathcal{H}T_\Phi(|\boldsymbol{k}-\boldsymbol{k_1}|\eta) + T_\Phi'(|\boldsymbol{k}-\boldsymbol{k_1}|\eta)\right)\Big]\ .$$

SIGWs: Power-spectrum and kernel function

The dimension-less power-spectrum $\Delta^2_{\gamma_{SS}}(k)$ is defined by

$$\langle {\gamma_{SS}}^{(\lambda)}(\boldsymbol{k},\eta){\gamma_{SS}}^{(\lambda')}(\boldsymbol{k}',\eta)\rangle = (2\pi)^3\delta_{\lambda\lambda'}\delta^3(\boldsymbol{k}+\boldsymbol{k}')\frac{2\pi^2}{k^3}\Delta^2_{\gamma_{SS}}(k)\,.$$

For each polarisation λ of SIGWs then ($v = k_1/k$, $u = |\boldsymbol{k}-\boldsymbol{k}_1|/k, x = k\eta$)

$$\Delta^2_{\gamma_{SS}}(k) = 16k^2\int_0^\infty dv\int_{|v-1|}^{v+1} du\,\frac{v^2}{u^2}\left(1-\left(\frac{1+v^2-u^2}{2v}\right)^2\right)^2\Delta^2_\Phi(uk)\Delta^2_\Phi(vk) \times I^2_{SS}(x,u,v),$$

$I_{SS}(x,u,v)$ being the kernel function.

SIGWs: Kernel function

$$I_{SS}(x,u,v) = \int_0^x d\tilde{x}\, G(x,\tilde{x}) \left[T_\Phi(v\tilde{x}) T_\Phi(u\tilde{x}). \right.$$
$$\left. + \frac{1}{2} \left(\mathcal{H} T_\Phi(v\tilde{x}) + k \dot{T}_\Phi(v\tilde{x}) \right) \left(\mathcal{H} T_\Phi(u\tilde{x}) + k \dot{T}_\Phi(u\tilde{x}) \right) \right] ,$$

giving us an oscillation average

$$\langle I_{SS}^2(x,u,v) \rangle = \frac{3^4}{2^5 k^2} \left(\frac{1}{uvx} \right)^2 \left\{ \Theta \left(\frac{u+v}{\sqrt{3}} - 1 \right) \frac{9\pi^2}{4} s^4 \right.$$
$$\left. + \left(\frac{3s^2}{2} \ln \left| \frac{1+s}{1-s} \right| - 3s \right)^2 \right\},$$

where $2uv/3\, s = (v^2 + u^2)/3 - 1$.

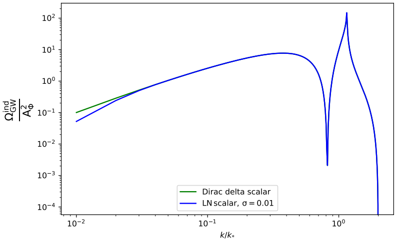
SIGWs: Power spectrum for peaked sources

Input: Dirac power spectrum: $\Delta^2_\Phi(k) = A_\Phi \, \delta\left(\ln \frac{k}{k_*}\right)$

$$\langle \Delta^{2(\lambda)}_{\gamma SS}(k) \rangle = 16 A_\Phi^2 \, (k_*/k)^2 \, \left[1 - \frac{k^2}{4k_*^2}\right]^2 \langle I_{SS}^2 \rangle_{u=v=k_*/k} \, \Theta(2k_* - k)$$

$$\langle I_{SS}^2 \rangle_{u=v=k_*/k} = \frac{3^6}{2^7 k^2} \frac{1}{x^2} \left(\frac{k}{k_*}\right)^4 \left\{ \frac{3^4 \pi^2}{2^4} \left(1 - \frac{2k_*^2}{3k^2}\right)^4 \left(\frac{k}{k_*}\right)^8 \Theta\left(\frac{2}{\sqrt{3}} - \frac{k}{k_*}\right) \right.$$
$$\left. + \left[-\left(\frac{2}{3} - \frac{k^2}{k_*^2}\right) + \frac{3}{4}\left(1 - \frac{2k_*^2}{3k^2}\right)^2 \left(\frac{k}{k_*}\right)^4 \ln\left|1 - \frac{4k_*^2}{3k^2}\right| \right]^2 \right\}$$

SIGWs: Power spectrum for peaked sources



$$\Omega_{\rm GW}^{\rm ind}(k,\eta) = \frac{1}{12}\left(\frac{k}{\mathcal{H}}\right)^2 \sum_{\lambda=+,\times} \langle \Delta_{\gamma ss}^{2(\lambda)}(k) \rangle$$

- Logarithmic resonance at $k/k_* = 2/\sqrt{3}$.
- The spectrum vanishes above $k/k_* = 2$.
- Zero point at $k/k_* = \sqrt{2/3}$.

SIGWs: Universal IR behaviour

- Cai *et al.* 2020: For a scalar power spectrum which is peaked at k_*, whose width $\Delta \ll 1$, $\Omega_{GW} \propto k^3$ for generation in radiation domination $\rightarrow$ not enough to label as SIGWs
- Yuan *et al.* 2020: It has a correction $\Omega_{GW} \propto k^{3-4/\ln\frac{4k_*^2}{3k^2}}$ in IR, $\Omega_{GW} \propto k^{2-4/\ln\frac{4k_*^2}{3k^2}}$ near the peak.
- This log-dependent slope can be taken as a template for distinguishing SIGWs from other sources.

Set-up: Scalar-tensor Induced GWs

- We consider the same space-time in radiation domination

$$ds^2 = -e^{2\Phi} dt^2 + a^2 e^{-2\Psi} \left(e^{\gamma}\right)_{ij} dx^i dx^j \,,$$

- $A_\gamma < A_\Phi \to$ tensor-tensor induced GWs can be ignored.
- Equation for scalar-tensor induced GWs (no anisotropic stress $\to \Phi = \Psi$)

$$\gamma_{ij}'' + 2\mathcal{H}\gamma_{ij}' - \nabla^2\gamma_{ij} = 4\Phi\nabla^2\gamma_{ij} + 4\Phi'\gamma_{ij}' \,.$$

Spectrum of scalar-tensor induced GWs

For the right and left polarizations of the induced GWs respectively
($v = k_1/k,\ u = |\boldsymbol{k} - \boldsymbol{k}_1|/k$),

$$\Delta^2_{\gamma_1,\,\mathrm{R/L}}(k) = \frac{1}{32}\int_0^\infty dv \int_{|v-1|}^{v+1} \frac{du}{v^6 u^2}\,\Delta^2_\Phi(uk)\,\overline{\mathcal{I}^2(x,u,v)}$$
$$\times\left[\left((v+1)^2 - u^2\right)^4 \Delta^2_{\gamma_0,\,\mathrm{R/L}}(vk) + \left((v-1)^2 - u^2\right)^4 \Delta^2_{\gamma_0,\,\mathrm{L/R}}(vk)\right],$$

where the kernel is defined as ($T \to$ transfer function)

$$\mathcal{I}(u,v) = \int_0^\infty k\,d\tilde{x}\,G(x,\tilde{x})\,\left[v^2\,T_\gamma(v\tilde{x})\,T_\Phi(u\tilde{x}) - \dot{T}_\gamma(v\tilde{x})\,\dot{T}_\Phi(u\tilde{x})\right].$$

Kernel behaviour

Oscillation average of $\mathcal{I}^2$ $(2uv/\sqrt{3}\,s = v^2 + u^2/3 - 1\ )$

$$\langle \mathcal{I}^2 \rangle = \frac{9}{2^7 x^2} \left(\frac{v}{u/\sqrt{3}} \right)^2 \times \left[\pi^2 (1-s^2)^2 \Theta\left(1 - |s|\right) + \left(2s + (1-s^2) \log \left| \frac{1+s}{1-s} \right| \right)^2 \right] .$$

- **IR limit:** $u \sim v \sim 1/k \gg 1 \longrightarrow s = \frac{2}{\sqrt{3}}$, c.f. $s = 1$ for SIGWs, leading to logarithmic running, which is absent here.
- **UV limit:** $v \to 1$ and $u \to 0 \longrightarrow s \sim u \longrightarrow \langle \Delta^2_{\gamma_1} \rangle \sim 1/u^4 \to$ divergence. $u \to 1$ and $v \to 0 \longrightarrow s \to 1/v \to$ no divergence.

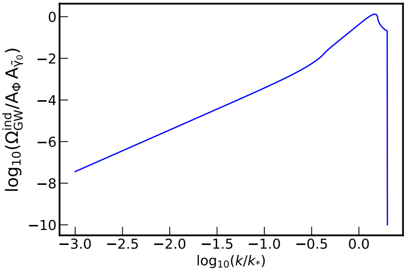
Kernel behaviour

- No zero point like SIGWs.
- No logarithmic resonance like SIGWs at $s = \pm 1\,(v = \pm(1 - u/\sqrt{3}))$.

  $\langle \mathcal{I}^2 \rangle = \frac{9}{2^5}\left(1 - \frac{\sqrt{3}}{u}\right)^2$

  This diverges in the $u \to 0$ limit.

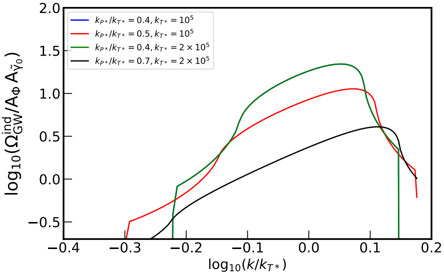
Monochromatic sources: $\Delta^2_{\Phi,\,\gamma_0}(k) = A_{\Phi,\,\gamma_0}\,\delta\left(\ln \frac{k}{k_{P,T*}}\right)$



Same peak location ($k_{P*} = k_{T*} = k_*$)

$$\langle\Delta^2_{\gamma_1}(k)\rangle = A_\Phi\,A_{\tilde\gamma_0}\,\left(\frac{k}{k_*}\right)^2\left[1+\frac{k^4}{16k_*^4}+\frac{3k^2}{2k_*^2}\right]\langle\mathcal{I}^2\rangle_{u=v=k_*/k}\,\Theta(2k_*-k)\,.$$

Power-spectrum: Non-chiral GWs



Different peak locations

$$\langle \Delta^2_{\gamma_1}(k)\rangle = A_\Phi\, A_{\tilde\gamma_0}\,\frac{k^2}{k_{P*}k_{T*}}\left[\frac{(k^2+k_{T*}^2-k_{P*}^2)^2}{k^2k_{T*}^2} + \frac{(4k^2k_{T*}^2+(k^2+k_{T*}^2-k_{P*}^2)^2)^2}{16k^4k_{T*}^4}\right] \times \langle \mathcal{I}^2\rangle_{v=k_{T*}/k,u=k_{P*}/k}\,\Theta(k_{P*}-|k_{T*}-k|)\,\Theta(k_{T*}+k-k_{P*})\,.$$

Power-spectrum: Non-chiral GWs

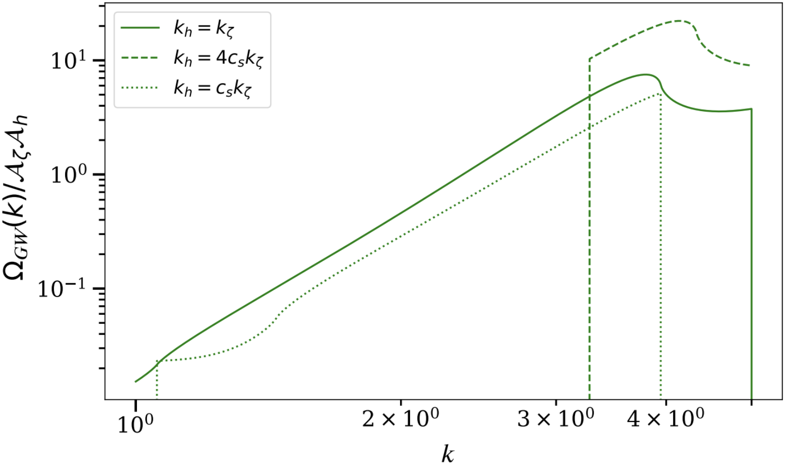
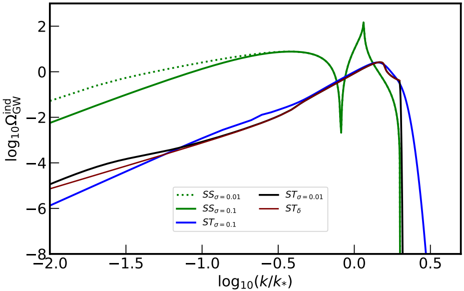


Figure: Spectral density of GWs induced by scalar-tensor modes peaking at different locations(Picard *et al.* 2024)

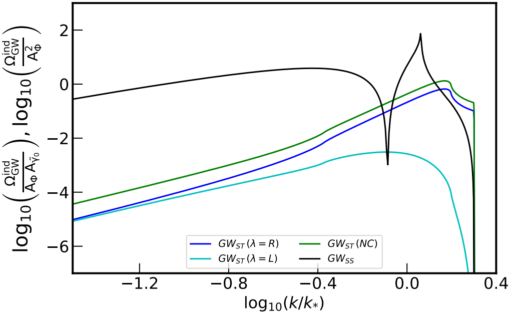
Power-spectrum: Non-chiral GWs

Log-normal sources $\Delta^2_{\Phi/\gamma_0}(k) = \frac{A_{\Phi/\gamma_0}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2} \ln^2 \frac{k}{k_*}\right)$



- Amplitude not very sensitive to σ, only the shape changes.
- Close to peak $\rightarrow$ converge to the monochromatic scenario.
- Increase of σ $\rightarrow$ no sharp cut-off.

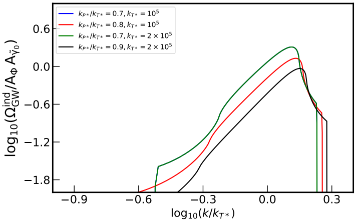
Monochromatic sources: same peak location

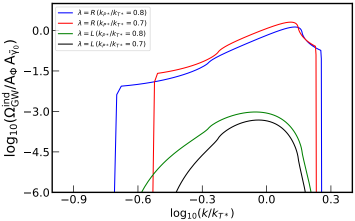


$$\Omega_{\rm GW}^{\rm ind}(k,\eta) = \frac{1}{768} A_\Phi\, A_{\tilde\gamma_0(\rm R/L)} \left(\frac{k}{k_*}\right)^6 \left[2 + 32\left(\frac{k_*}{k}\right)^4 + 48\left(\frac{k_*}{k}\right)^2\right]$$
$$\times \langle \mathcal{I}^2 \rangle_{u=v=k_*/k}\, \Theta(2k_* - k)\,.$$

Power-spectrum: Chiral GWs

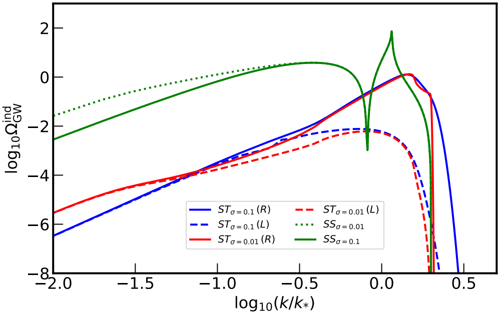
Monochromatic sources: different peak locations





Power-spectrum: Chiral GWs

Log-normal sources



Future prospects

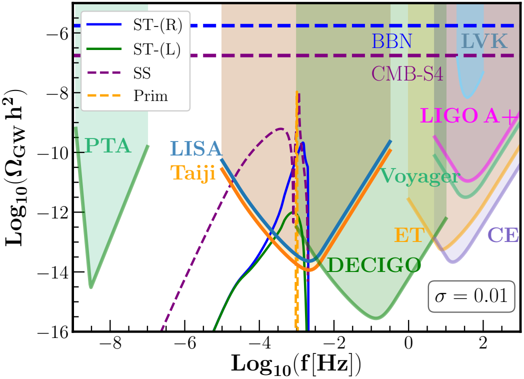


Figure: PGWs, SIGWs, and STGWs with $A_{\gamma_{0,\mathrm{R}}} = 10^{-3}$, $A_{\Phi} = 10^{-2}$, $f_* = 10^{-3}\,\mathrm{Hz}$, with the sensitivity curves of detectors.

Future prospects (contd.)

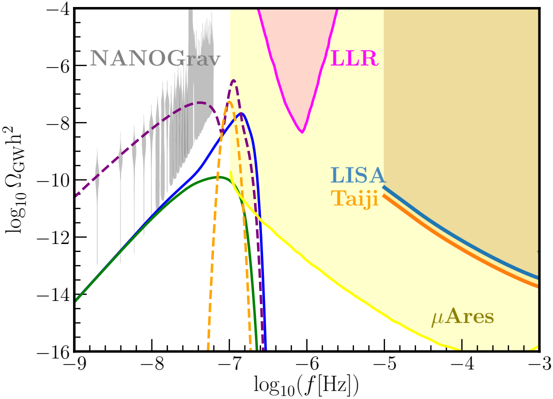


Figure: PGWs, SIGWs, and STGWs with $A_{\gamma_{0,\rm R}} = 10^{-2}$, $A_\Phi = 10^{-1}$, $\sigma = 0.1$, $f_* = 10^{-7}\,\rm Hz$, with the sensitivity curves of detectors, and NANOGrav results.

Summary and Outlook

- Our study explores non-linear interactions between **scalar and tensor perturbations**, modifying existing GWs.
- **Chirality** can be used as a **distinguishing characteristic**.
- We have to **mitigate the UV divergence**.

  $\rightarrow$ problematic term: $\Phi\nabla^2\gamma_{ij} \rightarrow$ include higher order term, i.e. $\Phi^{(1)2}\nabla^2\gamma_{ij}$ and $\Phi^{(1)}\nabla^2\gamma^{(2)}_{ij\,(\Phi\gamma)}$
- Investigate the scenario with **non-Gaussian** primordial perturbations.
- Analyze the findings in light of **recent PTA results**.

Thank You